Solutions must be typeset in LaTeX. Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. Any collaboration should involve only discussion; all writing **must** be done individually. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1. Consider sampling $X_1, \ldots, X_n \sim \{\pm 1\}$ (independently) and define $S_n = X_1 + \cdots + X_n$. Prove that $\mathbb{E}|S_n| = \Theta(\sqrt{n})$.

N.b. The truth is that $\mathbb{E}|S_n| = n\binom{n-1}{\lfloor (n-1)/2 \rfloor}/2^{n-1} = \left(\sqrt{2/\pi} + o(1)\right)\sqrt{n}$, but you do not need to prove this (though you are certainly welcome to do so!).

Note: There are tons of ways to prove this. The easiest I've found uses Jensen for the upper-bound and Markov+Stirling for the lower-bound.

Problem 2. A three-term arithmetic progression (3-AP) is a set of the form $\{x, x+d, x+2d\}$ where x is an integer and d is a positive integer. Prove that there is a subset $S \subseteq [n]$ with $|S| \geq \Omega(\sqrt{n})$ which has no 3-AP.

N.b. There is a clever construction of such a subset of size $\geq ne^{-O(\log^{1/2} n)}$, which has recently been proved to be almost optimal in the sense that the largest such subset has size $\leq ne^{-\Omega(\log^{\alpha} n)}$ for some $\alpha > 0$.

Problem 3. Let H be a k-uniform hypergraph. Prove that if H has n vertices and average degree $d \geq 1$, then

$$\alpha(H) \geq \left(1 - \frac{1}{k}\right)\frac{n}{d^{1/(k-1)}}.$$

Problem 4. A set of points $P \subseteq \mathbb{R}^n$ is said to be *acute* if every triangle formed by a triple of points in P is acute. Note that a degenerate triangle (all three points on a line) is considered obtuse and that a right triangle is not acute. It is still an open question as to the largest size of an acute set: the best lower- and upper-bounds to-date are $2^{n-1} + 1$ and $2^n - 1$, respectively. The upper-bound uses a clever volume argument which we won't discuss (though feel free to inquire in office hours!). The lower-bound was actually outlined on a forum of all places and (I believe) no one knows the author's true identity!

This problem exists to walk you through the very first exponential lower-bound established on this problem.

A. A triple of subsets (A, B, C) is said to be *bad* if A, B, C are distinct and $A \cap B \subseteq C \subseteq A \cup B$. Prove that there is some absolute constant $c > 1$ and a family $\mathcal{F} \subseteq 2^{[n]}$ with $|\mathcal{F}| \geq \Omega(c^n)$ which contains no bad triple.

B. Use part A. to construct an acute set $P \subseteq \mathbb{R}^n$ with $|P| \geq \Omega(c^n)$ where $c > 1$ is some absolute constant.

Problem 5 (Challenge)**.** Let G be any graph with no isolated vertices. Prove that G contains an *induced* subgraph H on $\geq \alpha(G)/2$ many vertices where every vertex of H has odd degree.

N.b. This is asymptotically tight, as witnessed by the vertex-edge incidence graph of K_n.

Hint: Consider a maximum independent set I along with a minimal set which contains at least one neighbor of every vertex in I.