Solutions must be typeset in LaTeX. Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. Any collaboration should involve only discussion; all writing **must** be done individually. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1. Suppose that $G_1, \dots, G_k$ are all graphs on the same vertex-set V and set $m_i = e(G_i)$. Prove that there is a common bipartition of V such that G_i has at least $\frac{m_i}{2} - \sqrt{km_i}$ many edges crossing this bipartition for each $i \in [k]$.

Problem 2. A k-AP is a set of the form $\{x, x+d, x+2d, \dots, x+(k-1)d\}$ where x is an integer and d is a positive integer. Consider the random subset A of $[n]$ where each element is independently included in A with probability p. For each fixed $k \geq 3$, find a threshold probability for the event that A contains a k-AP. That is, locate some function $t = t(n)$ such that, as $n \to \infty$,

$$\mathbf{Pr}[A \text{ contains a } k\text{-AP}] \to \begin{cases} 0 & \text{if } p \ll t, \\ 1 & \text{if } p \gg t. \end{cases}$$

Problem 3 (Unbalancing lights revisited)**.** In class, we proved that there is a constant $c > 0$ such that for every matrix $A \in \{\pm 1\}^{n \times n}$, we can locate $x, y \in \{\pm 1\}^n$ such that $\left|\sum_{i,j} A_{i,j} x_i y_j\right| \geq cn^{3/2}$. Prove that this bound is optimal.

Explicitly: prove that there is a constant C so that for every positive integer n, one can find a matrix $A \in \{\pm 1\}^{n \times n}$ with the property that $\left|\sum_{i,j} A_{i,j} x_i y_j\right| \leq Cn^{3/2}$ for every $x, y \in \{\pm 1\}^n$.

Problem 4 (Existence of thresholds)**.** Fix a non-empty finite set Ω and let Ω_p denote a random subset of Ω where each element is included independently with probability p.[1] A property $\mathcal{P} \subseteq 2^\Omega$ is said to be *increasing* if $A \in \mathcal{P} \land A \subseteq B \implies B \in \mathcal{P}$ and is said to be *non-trivial* if $\mathcal{P} \notin \{\varnothing, 2^\Omega\}$.

A. Fix an increasing property $\mathcal{P}$. Prove that the function $p \mapsto \mathbf{Pr}[\Omega_p \in \mathcal{P}]$ is increasing and continuous.

   Hint: Some "coupling" idea is necessary here. Perhaps the easiest is to consider a uniformly random function $\phi\colon \Omega \to [0,1]$ and define $A_p = \{x \in \Omega : \phi(x) \leq p\}$ for each $p \in [0,1]$.

B. Fix an increasing property $\mathcal{P}$, a number $p \in [0,1]$, and a positive integer m. Prove that $\mathbf{Pr}[\Omega_p \notin \mathcal{P}] \leq \mathbf{Pr}[\Omega_{p/m} \notin \mathcal{P}]^m$.

   Hint: Consider the union of m independent copies of $\Omega_{p/m}$.

C. Fix a finite set $\Omega = \Omega(n)$ and a non-trivial, increasing property $\mathcal{P} = \mathcal{P}(n)$ (really, a sequence of sets and properties). Prove that there is a function $t = t(n)$ such that, as $n \to \infty$,

$$\mathbf{Pr}[\Omega_p \in \mathcal{P}] \to \begin{cases} 0 & \text{if } p \ll t, \\ 1 & \text{if } p \gg t. \end{cases}$$

   That is, prove that there is a threshold probability for the event $\mathcal{P}$.

   Hint: Pick t such that $\mathbf{Pr}[\Omega_t \in \mathcal{P}] = 1/2$ (why can you do this?).

[1] $G(n,p)$ corresponds to Ω_p when $\Omega = \binom{[n]}{2}$.