Problem 1. Recall that Chebyshev's inequality implied that if X is a non-negative random variable, then $\mathbf{Pr}[X = 0] \le \frac{\mathbf{Var}\, X}{(\mathbb{E}\, X)^2}$. Improve this inequality slightly to

$$\mathbf{Pr}[X = 0] \le \frac{\mathbf{Var}\, X}{\mathbb{E}(X^2)}.$$

Hint: Cauchy–Schwarz states that $(\mathbb{E}\, XY)^2 \le \mathbb{E}\, X^2 \cdot \mathbb{E}\, Y^2$.

Problem 2. Fix $\epsilon > 0$. Say that a triangle is *almost-equilateral* if all angles in the triangle are within $(60 \pm \epsilon)^\circ$. Prove that there is some absolute constant $c = c(\epsilon) > 0$ such that for every n, there is a set $P \subseteq \mathbb{R}^n$ with $|P| \ge c^n$ with the property that every triangle formed by a triple of points in P is almost equilateral.

Hint: Show that if $x, y, z \sim \{0,1\}^n$ (independent), then $\|x - y\|^2$ is likely very close to $n/2$ and $\langle x - z, y - z \rangle$ is likely very close to $n/4$.

Problem 3. Improve Chernoff's bound for sums of independent Bernoulli random variables with small means.

A. Suppose that X is a random variable satisfying $0 \le X \le 1$. Prove that $\mathbb{E}\, e^{tX} \le e^{(e^t - 1)\,\mathbb{E}\, X}$ for every $t \in \mathbb{R}$.

B. Suppose that $X_1, \ldots, X_n$ are independent random variables satisfying $0 \le X_i \le 1$ for every i. Set $S = X_1 + \cdots + X_n$ and $\mu = \mathbb{E}\, S$. Prove that

$$\mathbf{Pr}[S \ge (1+\delta)\mu] \le e^{-\frac{\delta^2}{2+\delta}\mu} \qquad \text{for all } \delta > 0,$$
$$\mathbf{Pr}[S \le (1-\delta)\mu] \le e^{-\frac{\delta^2}{2}\mu} \qquad \text{for all } 0 < \delta < 1.$$

Hint: The following inequalities may be helpful:

$$\log(1+x) \ge \frac{2x}{2+x} \quad \text{for all } x > 0, \qquad \text{and} \qquad \log(1-x) \ge -\frac{x(2-x)}{2(1-x)} \quad \text{for all } 0 < x < 1.$$

C. Consider $G(n,p)$ with $p = \frac{c}{n}$ for some fixed $c > 0$. Use the Chernoff bound from part B. to prove the following facts:

- The average degree of $G(n, \frac{c}{n})$ is $c + o(1)$ w.h.p.
- $\Delta\big(G(n, \frac{c}{n})\big) \le O(\log n)$ w.h.p.

Can the Chernoff bound proved in class be used to establish either of these facts?