These solutions are from http://math.cmu.edu/~cocox/teaching/discrete20/quiz11sol.pdf

Problem 1. Let X be a random variable on a finite or countable probability space $(\Omega, \mathbf{Pr})$ such that $\mathbb{E}\, X$ is finite. Show that there exist $\omega, \omega' \in \Omega$ for which $X(\omega) \leq \mathbb{E}\, X \leq X(\omega')$.

Solution. Suppose for the sake of contradiction that $X(\omega) > \mathbb{E}\, X$ for all $\omega \in \Omega$. Then

$$\mathbb{E}\, X = \sum_{\omega \in \Omega} X(\omega)\, \mathbf{Pr}[\omega] > \sum_{\omega \in \Omega} (\mathbb{E}\, X)\, \mathbf{Pr}[\omega] = \mathbb{E}\, X;$$

a contradiction. Thus, there is some $\omega \in \Omega$ such that $X(\omega) \leq \mathbb{E}\, X$. A symmetric argument shows that there is some $\omega' \in \Omega$ for which $X(\omega') \geq \mathbb{E}\, X$. □

Problem 2. State and prove Markov's inequality.

Solution. Markov's inequality states that if X is a non-negative random variable and $t > 0$, then

$$\mathbf{Pr}[X \geq t] \leq \frac{1}{t}\, \mathbb{E}\, X.$$

To prove this, let $A = \{\omega \in \Omega : X(\omega) \geq t\}$; then, by using $X(\omega) \geq t$ for $\omega \in A$ and $X(\omega) \geq 0$ for $\omega \in \overline{A}$, we bound

$$\begin{aligned}\mathbb{E}\, X &= \sum_{\omega \in \Omega} X(\omega)\, \mathbf{Pr}[\omega] = \sum_{\omega \in A} X(\omega)\, \mathbf{Pr}[\omega] + \sum_{\omega \in \overline{A}} X(\omega)\, \mathbf{Pr}[\omega] \\ &\geq t \cdot \sum_{\omega \in A} \mathbf{Pr}[\omega] = t \cdot \mathbf{Pr}[A] = t \cdot \mathbf{Pr}[X \geq t],\end{aligned}$$

implying that $\mathbf{Pr}[X \geq t] \leq \frac{1}{t}\, \mathbb{E}\, X$ since $t > 0$. □

Problem 3. State and prove Chebyshev's inequality.

Solution. Chebyshev's inequality states that if X is a random variable for which $\mathbb{E}\, X$ is finite, then for any $t > 0$,

$$\mathbf{Pr}\big[|X - \mathbb{E}\, X| \geq t\big] \leq \frac{\mathbf{Var}\, X}{t^2}.$$

To prove this, set $Y = \big(X - \mathbb{E}\, X\big)^2$, which is well-defined since $\mathbb{E}\, X$ is finite. Since Y is non-negative and $t > 0$, we can use Markov's inequality to bound

$$\mathbf{Pr}\big[|X - \mathbb{E}\, X| \geq t\big] = \mathbf{Pr}[Y \geq t^2] \leq \frac{1}{t^2}\, \mathbb{E}\, Y = \frac{1}{t^2}\, \mathbb{E}\big(X - \mathbb{E}\, X\big)^2 = \frac{\mathbf{Var}\, X}{t^2}.$$

□

Problem 4. Suppose a coin is biased so that $\mathbf{Pr}[H] = p$ and $\mathbf{Pr}[T] = 1 - p$ for some fixed $p \in (0, 1)$. Consider repeatedly flipping this coin (flips are independent) until we see a T. Let X be the random variable which counts the number of heads in the experiment (e.g. $X(HHHT) = 3$ and $X(T) = 0$). Compute $\mathbb{E}\, X$.

Solution. Since the coin flips are independent, we see that for any $n \in \mathbb{N}$, we have

$$\mathbf{Pr}[X = n] = \mathbf{Pr}[H^n T] = p^n(1-p).$$

Thus, since $p \in (0,1)$,

$$\begin{aligned}
\mathbb{E}\, X &= \sum_{n\geq 0} n \cdot \mathbf{Pr}[X = n] = (1-p)\sum_{n\geq 0} n \cdot p^n = (1-p)\sum_{m\geq 1}\sum_{n\geq m} p^n \\
&= (1-p)\sum_{m\geq 1} p^m \sum_{n\geq 0} p^n = (1-p)\sum_{m\geq 1}\frac{p^m}{1-p} = \sum_{m\geq 1} p^m = \frac{p}{1-p}.
\end{aligned}$$

Here's another way to compute $\mathbb{E}\, X$:

$$\begin{aligned}
\mathbb{E}\, X &= \sum_{n\geq 0} n \cdot \mathbf{Pr}[X = n] = (1-p)\sum_{n\geq 0} n \cdot p^n = p(1-p)\sum_{n\geq 0}(n+1)p^n \\
&= p(1-p)\left[\frac{d}{dx}\sum_{n\geq 0} x^{n+1}\right]_{x=p} = p(1-p)\left[\frac{d}{dx}\frac{x}{1-x}\right]_{x=p} \\
&= p(1-p)\frac{1}{(1-p)^2} = \frac{p}{1-p}.
\end{aligned}$$

And one more for good measure. For $\omega \in \Omega$, let ω_i denote the value of the ith coin flip in ω (if it exists).

$$\begin{aligned}
\mathbb{E}\, X &= \sum_{\omega\in\Omega} X(\omega)\,\mathbf{Pr}[\omega] = \sum_{\omega\in\Omega:\ \omega_1 = T} X(\omega)\,\mathbf{Pr}[\omega] + \sum_{\omega\in\Omega:\ \omega_1 = H} X(\omega)\,\mathbf{Pr}[\omega] \\
&= 0 + \sum_{\omega\in\Omega:\ \omega_1 = H} X(H\omega_2\cdots)\,\mathbf{Pr}[H\omega_2\cdots] = \sum_{\omega\in\Omega} X(H\omega)\,\mathbf{Pr}[H\omega] \\
&= \sum_{\omega\in\Omega}\big(X(\omega)+1\big)\cdot p \cdot \mathbf{Pr}[\omega] = p \cdot \sum_{\omega\in\Omega} X(\omega)\,\mathbf{Pr}[\omega] + p \cdot \sum_{\omega\in\Omega}\mathbf{Pr}[\omega] \\
&= p \cdot \mathbb{E}\, X + p,
\end{aligned}$$

so since $p \in (0,1)$, $\mathbb{E}\, X = \frac{p}{1-p}$. □

Problem 5 (Bonus)**.** Recall property S_k from HW7(1). Let $n(k)$ denote the least integer n for which there is a tournament with n teams which has property S_k. Prove that

$$2^{k+1} - 1 \leq n(k) \leq k^2 2^{k+2}.$$

Solution. It is quick to verify $n(1) = 3$, which satisfies both bounds. Indeed, clearly $n(1) > 2$ and the example in HW7(1) shows $n(1) \leq 3$. Hence, throughout we assume $k \geq 2$.

Upper bound. In HW7(1) we showed that if $\binom{n}{k}(1-2^{-k})^{n-k} < 1$, then there exists a tournament with n teams which has property S_k. In other words, if $\binom{n}{k}(1-2^{-k})^{n-k} < 1$, then $n(k) \leq n$. Thus, it suffices to show that if $n = k^2 2^{k+2}$, then $\binom{n}{k}(1-2^{-k})^{n-k} < 1$. To see this,

$$\begin{aligned}
\binom{n}{k}(1-2^{-k})^{n-k} &\leq n^k e^{-(n-k)2^{-k}} = \exp\big\{k\log n + k2^{-k} - n2^{-k}\big\} \\
&= \exp\big\{k\log(k^2 2^{k+2}) + k2^{-k} - 4k^2\big\} \\
&< \exp\left\{2k\log k + \frac{9}{4}k - 3k^2\right\} < \exp\big\{3k - 2k^2\big\} < 1,
\end{aligned}$$

for all $k \geq 2$.

Lower bound. We proceed by induction on k with the base case of $n(1) = 3$.

Let T be a fixed tournament with teams V and property S_k where $|V| = n(k)$. For $v \in V$, let $N^+(v)$ denote the set of teams that v beat and let $N^-(v)$ denote the set of teams that beat v.

Fix any $v \in V$, set $V' = N^-(v)$ and let T' be the tournament induced on V'. We claim that T' has property S_{k-1}. Indeed, fix any $S' \in \binom{V'}{k-1}$ and set $S = S' \cup \{v\}$. Since $|S| = k$ and T has property S_k, there must be some other team $u \in V$ which beats all teams in S; that is $S \subseteq N^+(u)$. We claim that $u \in V'$. Indeed, if $u \notin V'$, then $u \in N^+(v)$ (since $u \neq v$) which would mean that team v beat team u; a contradiction. Thus, $u \in V'$, and so there is a team in tournament T' which beats all teams in S'. Since $S' \in \binom{V'}{k-1}$ was arbitrary, this means that T' has property S_{k-1}; in particular $|N^-(v)| = |V'| \geq n(k-1)$.

Now, for any $u \neq v \in V$, either $v \in N^-(u)$ or $u \in N^-(v)$ (but not both). Hence,

$$\binom{|V|}{2} = \sum_{\{u,v\} \in \binom{V}{2}} 1 = \sum_{v \in V} \sum_{u \in N^-(v)} 1 = \sum_{v \in V} |N^-(v)| \geq |V| \cdot n(k-1)$$

$$\implies \frac{|V| - 1}{2} \geq n(k-1).$$

Thus, by induction, we have $n(k) = |V| \geq 2n(k-1) + 1 \geq 2(2^k - 1) + 1 = 2^{k+1} - 1$. $\square$