These solutions are from http://math.cmu.edu/~cocox/teaching/discrete20/quiz12sol.pdf

Problem 1. Prove that G is a connected, 2-regular graph if and only if G is a cycle.

Solution. Certainly every cycle is connected and 2-regular, so we need only show the "only if" direction.

By HW8(5a), we know that G has a cycle $(v_1, \dots, v_k)$. Since G is 2-regular, we observe that there cannot be any additional edges among $v_1, \dots, v_k$. Hence, if $V = \{v_1, \dots, v_k\}$, then G is a cycle.

Suppose this is not the case; that is, $V \setminus \{v_1, \dots, v_k\} \neq \varnothing$. Since G is connected, we know that there must be an edges between $\{v_1, \dots, v_k\}$ and $V \setminus \{v_1, \dots, v_k\}$; suppose such an edge is $\{v_j, u\}$. But then v_{j-1}, v_{j+1}, u are distinct vertices adjacent to v_j, meaning $\deg(v_j) \geq 3$; a contradiction. □

Problem 2. There are $n \geq 3$ participants in an event. Each of these participants know at least $n/2$ other participants. Show that there is a way to seat the participants around a circular table so that each participant knows both people seated next to them.

Solution. Build a graph $G = (V, E)$ where V is the set of participants and for $u \neq v \in V$, $\{u, v\} \in E$ if u and v know one another. By assumption, G is a graph on $n \geq 3$ vertices and $\delta(G) \geq n/2$. Dirac's theorem then tells us that G has a Hamilton cycle. Thus, seating the participants around the table in the order that they're visited by such a Hamilton cycle yields a valid arrangement. □

Problem 3. Let G be any graph. A *cycle decomposition* of G is a collection of cycles $C_1, \dots, C_k$ that partition the edge-set of G; that is $E = \bigsqcup_{i=1}^{k} E(C_i)$. Note that in a cycle decomposition, the cycles can share vertices, but they cannot share edges.

Show that G has a cycle decomposition if and only if every vertex of G has even degree.

Solution. Suppose that G has a cycle decomposition $C_1, \dots, C_k$. Fix any vertex $v \in V$ and let $E_v = \{e \in E : v \in e\}$. Observe that for any $i \in [k]$, $|E_v \cap E(C_i)| \in \{0, 2\}$. Hence, since $C_1, \dots, C_k$ form a partition of E, we have

$$\deg(v) = |E_v| = \sum_{i=1}^{k} |E_v \cap E(C_i)|,$$

which is even.

Now, suppose that every vertex of G has even degree. We prove that G has a cycle decomposition by induction on the number of edges of G. Observe that if $|E| = 0$, then this is vacuously true, so suppose that $|E| \geq 1$.

We can decompose G into its connected components $G = G_1 \cup \cdots \cup G_\ell$, so G_i is connected and the G_i's are vertex disjoint. Since $|E| \geq 1$, by relabeling the G_i's if necessary, we may suppose that $|E(G_1)| \geq 1$; in particular, $\deg(v) \geq 1$ for all $v \in V(G_1)$ (why?). By assumption, each vertex of G has even degree, so in fact $\deg(v) \geq 2$ for all $v \in V(G_1)$. Hence, thanks to HW8(5a), we know that G_1 has a cycle; call it C_1.

Form a new graph G' by deleting the edges of C_1 from G. Since every $v \in V$ is incident to either 0 or 2 edges in C_1, we see that every vertex of G' also has even degree. Furthermore, $|E(G')| = |E| - |E(C_1)| < |E|$, so by induction we can find a cycle decomposition of G', call it $C_2, \ldots, C_k$ (note that this may be empty if G' is the empty graph). By construction, $E = E(G') \sqcup E(C_1)$, so $E = \bigsqcup_{i=1}^{k} E(C_i)$. In other words, $C_1, \ldots, C_k$ is a cycle decomposition of G. □

Problem 4. Let G be a graph. Let $\mathrm{conn}(G)$ denote the set of connected components of G (e.g. $\mathrm{conn}(G) = \{G\}$ if and only if G is connected). For a subset $U \subseteq V$, let $G - U$ denote the graph formed by deleting the vertices in U from G: formally, $V(G-U) = V \setminus U$ and $E(G-U) = E \cap \binom{V \setminus U}{2}$.

Show that if G is Hamiltonian, then $|\mathrm{conn}(G-U)| \leq |U|$ for all non-empty $U \subseteq V$.

Solution. Let $U \subseteq V$ be a non-empty subset of the vertices. If $U = V$, then $\mathrm{conn}(G-U) = \varnothing$, which certainly satisfies the condition, so suppose that $U \subsetneq V$.

Fix any Hamilton cycle $(v_1, \ldots, v_n)$; without loss of generality, we may suppose that $v_n \in U$. For $H \in \mathrm{conn}(G-U)$, let i be the largest index for which $v_i \in V(H)$ and define $f(H) = v_{i+1}$. We claim that f is an injection from $\mathrm{conn}(G-U)$ to U, which will imply the claim.

Firstly, we must argue that f is well-defined. Since $(v_1, \ldots, v_n)$ is a Hamilton cycle, for any $H \in \mathrm{conn}(G-U)$, there must be some i for which $v_i \in V(H)$ and thus there must also be a largest such i. Note that $i \in [n-1]$ since $v_n \in U$ and $V(H) \cap U = \varnothing$. Now, since i is the largest index for which $v_i \in V(H)$, we know that $v_{i+1} \notin V(H)$. Since H is a connected component of $G - U$ and $\{v_i, v_{i+1}\} \in E$, this means that $v_{i+1} \in U$ (why?). Thus, $f\colon \mathrm{conn}(G-U) \to U$ is well-defined.

Now, suppose that $H, R \in \mathrm{conn}(G)$ have $f(H) = f(R) = u \in U$. Since $(v_1, \ldots, v_n)$ is a Hamilton cycle, there is a unique $i \in [n]$ for which $u = v_i$. Then by definition, we must have $v_{i-1} \in V(H) \cap V(R)$, and so $H = R$ (why?). Thus f is an injection as desired. □

Problem 5 (Bonus)**.** Let G be a graph. For a subset $A \subseteq V$ and a vertex $v \in V$, define $\deg_A(v) = \bigl|\{u \in A : \{u, v\} \in E\}\bigr|$. Consider the following algorithm whose input is a graph $G = (V, E)$:

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procedure BiPartition(G)
    V_0 ← V
    V_1 ← ∅
    while there exists v ∈ V_i such that deg_{V_{1-i}}(v) < deg(v)/2 do
        V_i ← V_i \ {v}
        V_{1-i} ← V_{1-i} ∪ {v}
    end while
    return (V_0, V_1)
end procedure
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Prove the following:

1. BiPartition(G) eventually terminates and returns a pair (V_0, V_1) where $V = V_0 \sqcup V_1$.

   (Hint: Show the algorithm terminates after at most $|E|$ iterations of the while loop)

2. If BiPartition$(G) = (V_0, V_1)$, then G has at least $|E|/2$ edges between V_0 and V_1.

(Note: This yields a polynomial-time algorithm to find the subgraph in HW7(5))

Solution. Throughout the following, for sets $A, B \subseteq V$ with $A \cap B = \varnothing$, let $e[A, B]$ denote the number of edges of G with one vertex in A and the other in B. Observe that

$$e[A, B] = \sum_{v \in A} \deg_B(v) = \sum_{v \in B} \deg_A(v).$$

1. For $i \in \{0, 1\}$ and $t \geq 0$, let V_i^t denote the value of V_i *after* the t'th iteration of the while loop; so $V_0^0 = V$ and $V_1^0 = \varnothing$. We prove the following:

   (a) $V = V_0^t \sqcup V_1^t$.

   (b) If there is some $v \in V_i^t$ such that $\deg_{V_{1-i}^t}(v) < \deg(v)/2$, then $e[V_0^{t+1}, V_1^{t+1}] > e[V_0^t, V_1^t]$.

   By item (b), we know that $t \leq e[V_0^t, V_1^t] \leq |E|$, so the algorithm must terminate after at most $|E|$ iterations of the while loop, and will thus return a pair (V_0, V_1) which is a partition of V by item (a).

   For item (a), this is trivially true for $t = 0$, so we proceed by induction on t. For $t \geq 1$, we know that there is some $v \in V_i^{t-1}$ for which $V_i^t = V_i^{t-1} \setminus \{v\}$ and $V_{1-i}^t = V_{1-i}^t \cup \{v\}$. Since $V = V_0^{t-1} \sqcup V_1^{t-1}$ by the induction hypothesis, this implies that $V = V_0^t \sqcup V_1^t$.

   Now for item (b). By relabeling if necessary, we may suppose that $v \in V_0^t$; thus $V_0^{t+1} = V_0^t \setminus \{v\}$ and $V_1^{t+1} = V_1^t \cup \{v\}$. Since $V = V_0^t \sqcup V_1^t$ by item (a), we know that $\deg(v) = \deg_{V_0^t}(v) + \deg_{V_1^t}(v)$; in particular, since $\deg_{V_1^t}(v) < \deg(v)/2$, we see that $\deg_{V_0^t}(v) > \deg_{V_1^t}(v)$. Finally, since $V = V_0^t \sqcup V_1^t = V_0^{t+1} \sqcup V_1^{t+1}$, we can now calculate

$$\begin{aligned}
e[V_0^t, V_1^t] &= \sum_{u \in V_0^t} \deg_{V_1^t}(u) = \deg_{V_1^t}(v) + e[V_0^t \setminus \{v\}, V_1^t] \\
&< \deg_{V_0^t}(v) + e[V_0^t \setminus \{v\}, V_1^t] = \deg_{V_0^{t+1}}(v) + e[V_0^{t+1}, V_1^{t+1} \setminus \{v\}] \\
&= \deg_{V_0^{t+1}}(v) + \sum_{u \in V_1^{t+1} \setminus \{v\}} \deg_{V_0^{t+1}}(u) = e[V_0^{t+1}, V_1^{t+1}].
\end{aligned}$$

2. If $\text{BiPartition}(G) = (V_0, V_1)$, then we know that for each $i \in \{0, 1\}$ and each $v \in V_i$, we must have $\deg_{V_{1-i}}(v) \geq \deg(v)/2$. Since $V = V_0 \sqcup V_1$, we find that

$$2e[V_0, V_1] = \sum_{v \in V_0} \deg_{V_1}(v) + \sum_{v \in V_1} \deg_{V_0}(v) \geq \sum_{v \in V} \frac{\deg(v)}{2} = |E|,$$

   so $e[V_0, V_1] \geq |E|/2$.

□