These solutions are from http://math.cmu.edu/~cocox/teaching/discrete20/quiz13sol.pdf

Just so that there's no confusion, every graph in this quiz is assumed to be simple. That is, graphs cannot contain loops nor multiple edges between two vertices.

Problem 1. We say that a graph $G = (V, E)$ is 2-*edge-connected* if $G - e$ is connected for every $e \in E$. Show that if G is a connected graph wherein each vertex has even degree, then G is 2-edge-connected.

Solution. Suppose for the sake of contradiction that there such a graph G which is not 2-edge-connected. Let $e \in E$ be such that $G - e$ is not connected. Now, since G is connected and $G - e$ is not, we observe that $G - e$ has exactly two connected components, call then G_1, G_2, where $u \in V(G_1)$ and $v \in V(G_2)$ (why is this the case?).

Consider the graph G_1. Observe that $\deg_{G_1}(u) = \deg_G(u) - 1$ and that for any $w \in V(G_1) \setminus \{u\}$, $\deg_{G_1}(w) = \deg_G(w)$. This, however, means that G_1 has an odd number of odd-degree vertices, which we know to be impossible.

Here's an alternate proof: It is enough to show that for any $U \subseteq V$ with $U \notin \{\varnothing, V\}$, G has at least two edges between U and $V \setminus U$ (why is this enough?).

Fix any such U. Since G is connected, there is an edge $e \in E$ with one vertex in U and the other in $V \setminus U$. Suppose that $U \cap e = \{u\}$. Since G is connected and every vertex has even degree, G has an Eulerian walk. We may suppose this walk starts at u and traverses the edge e first: label the edges $e = e_1, e_2, \ldots, e_m$ in the order that they are traversed. Let $i \in [m]$ be the largest index for which $e_i \setminus U \neq \varnothing$. We claim that e_i is an edge between U and $V \setminus U$. Firstly, we know that $u \in e_m$, so if $i = m$, then e_m is an edge between U and $V \setminus U$ and we are done; thus suppose that $i \leq m - 1$. But now, by definition, we know that $e_{i+1} \subseteq U$, so since $|e_i \cap e_{i+1}| = 1$, we see that e_i has one vertex in U and the other in $V \setminus U$, as needed.

Finally, observe that $e_2 \setminus U \neq \varnothing$, so we know that $i \in \{2, \ldots, m\}$. In particular, e_1 and e_i are two distinct edges between U and $V \setminus U$. $\square$

Problem 2. Let $T = (V, E)$ be a tree on at least 2 vertices. Let $\ell(T)$ denote the number of leaves of T. Prove that

$$\ell(T) = 2 + \sum_{\substack{v \in V: \\ \deg(v) \geq 2}} \big(\deg(v) - 2\big).$$

Solution. Since T is a tree, we know that $|E| = |V| - 1$. In other words,

$$\begin{aligned} 1 = |V| - |E| = |V| - \frac{1}{2}\sum_{v \in V} \deg(v) &= \frac{1}{2}\sum_{v \in V}\big(2 - \deg(v)\big) \\ &= \frac{1}{2}\Bigg(\ell(T) + \sum_{\substack{v \in V: \\ \deg(v) \geq 2}} \big(2 - \deg(v)\big)\Bigg), \end{aligned}$$

which is equivalent to what we wanted to show. $\square$

Problem 3. Does there exist a planar graph G which is both triangle-free and has $\delta(G) \geq 4$?

Solution. No, such a graph does not exist.

Suppose for the sake of contradiction that such a graph G did exist. We first observe that we may suppose that G is connected. Indeed, if G were not connected, then we may decompose G into its connected components, each of which would, in turn, be a connected, planar, triangle-free graph with minimum degree at least 4. Hence, if no such connected graph exists, then no such graph exists at all.

Fix any embedding of G into the plane and let F denote the set of faces of G in this embedding. For $f \in F$, let $\ell(f)$ denote the length of the face f. Since $\delta(G) \geq 4$, we know that G is not just a single edge, so since G is triangle-free, we must have $\ell(f) \geq 4$ for all $f \in F$.

Now, applying the handshaking lemma, we find that

$$2|E| = \sum_{v \in V} \deg(v) \geq 4|V| \qquad \text{and} \qquad 2|E| = \sum_{f \in F} \ell(f) \geq 4|F|.$$

Since G is planar and connected, Euler's theorem implies

$$2 = |V| + |F| - |E| \leq \frac{1}{2}|E| + \frac{1}{2}|E| - |E| = 0;$$

a contradiction. □

Problem 4. Let G be a planar graph on n vertices. Prove that G has at most $3n$ edges.

Solution. We suppose first that G is connected. Fix any embedding of G into the plane and let F denote the set of faces of G in this embedding. For $f \in F$, let $\ell(f)$ denote the length of the face f.

If G is just a single edge, then the claim certainly holds, so we may suppose that G has at least two edges (and at least three vertices). Now, since G does not have multiple edges between any pair of vertices, this implies that every $f \in F$ has $\ell(f) \geq 3$. Hence, by the handshaking lemma, $2|E| = \sum_{f \in F} \ell(f) \geq 3|F|$.

Since G is planar and connected, Euler's formula tells us that

$$2 = |V| + |F| - |E| \leq n + \frac{2}{3}|E| - |E| = n - \frac{1}{3}|E| \implies |E| \leq 3n - 6 \leq 3n.$$

Lastly, let G be any planar graph on n vertices. Decomposing $G = G_1 \cup \cdots \cup G_k$ into its connected components and applying the previous fact, we have

$$|E| = \sum_{i=1}^{k} |E(G_i)| \leq \sum_{i=1}^{k} 3|V(G_i)| = 3n.$$

□

Problem 5 (Bonus)**.** For a positive integer k and a graph $G = (V, E)$, a coloring $\chi \colon V \to [k]$ is called a *proper k-coloring* if $\chi(u) \neq \chi(v)$ whenever $\{u, v\} \in E$ (i.e. adjacent vertices get different colors).

Prove that every triangle-free planar graph has a proper 4-coloring. (This is a special case of the famous four color theorem)

Solution. Suppose this is false. Since the property of being triangle-free and planar is preserved under taking subgraphs, we can consider a minimal counterexample to the claim; call it G. That is, G does not have a proper 4-coloring, but every proper subgraph of G does.

Let $v \in V$ and suppose for the sake of contradiction that $\deg(v) \leq 3$. Let $G' = G - v$ (the graph formed by deleting v and its incident edges). Since G' is a proper subgraph of G, there is a proper 4-coloring $\chi' \colon V(G') \to [4]$. Now, we observe that $|\{\chi'(u) : \{u, v\} \in E\}| \leq \deg(v) \leq 3$; hence, let $\chi \colon V \to [4]$ be the coloring where $\chi(u) = \chi'(u)$ for all $u \in V \setminus \{v\}$, and $\chi(v)$ is any color in $[4] \setminus \{\chi(u) : \{u, v\} \in E\}$. It is clear that χ is a proper 4-coloring of G, contradicting our original assumption.

Therefore, G is planar, triangle-free and has $\delta(G) \geq 4$. However, Problem 3 tells us that such a graph does not exist; i.e. every triangle-free planar graph has a proper 4-coloring. $\square$