These solutions are from http://math.cmu.edu/~cocox/teaching/discrete20/quiz14sol.pdf

Problem 1. Let $G = (V, E)$ be a connected graph and let $f : V \to X$ be any function (where X is any arbitrary set). Prove that either f is a constant function (i.e. $|f(V)| = 1$) or that there is an edge $\{u, v\} \in E$ where $f(u) \neq f(v)$. Show also that the connectivity assumption is crucial.
(Note: This fact pops up time and time again, so it's worth keeping in mind!)

Solution. Firstly, it is important that G is connected. If G is not connected, then there is some $U \subseteq V$ with both $U \neq \varnothing$ and $V \setminus U \neq \varnothing$ so that G has no edges between U and $V \setminus U$. We could then have the function $f : V \to \{1, 2\}$ where $f(v) = 1$ if $v \in U$ and $f(v) = 2$ otherwise. Then f is neither the constant function, nor is there any edge $\{u, v\} \in E$ for which $f(u) \neq f(v)$.

Here are two solutions, each using a different definition of connectivity.

Solution 1. Suppose that f is not a constant function; thus $|f(V)| \geq 2$. Fix any $x \in f(V)$ and define $U = \{u \in V : f(u) = x\}$. Since $x \in f(V)$, we know that $U \neq \varnothing$ and since $f(V) \neq \{x\}$, we know that $V \setminus U \neq \varnothing$. Since G is connected, there must be an edge between U and $V \setminus U$; suppose this edge is $\{u, v\} \in E$ where $u \in U$ and $v \in V \setminus U$. Then, by definition, $f(u) = x \neq f(v)$ as needed.

Solution 2. Suppose that f is not a constant function; thus there are $u, v \in V$ with $f(u) \neq f(v)$. Since G is connected, there must be a path $(u = x_1, x_2, \ldots, x_k = v)$ where $\{x_i, x_{i+1}\} \in E$ for all $i \in [k - 1]$. Let $\ell \in [k]$ be the largest index for which $f(x_\ell) = f(u)$; note that ℓ is well-defined since $f(x_1) = f(u)$. Furthermore, observe that $\ell \in [k - 1]$ since $f(x_k) = f(v) \neq f(u)$. In particular, $f(x_{\ell+1}) \neq f(u) = f(x_\ell)$ and so $\{x_\ell, x_{\ell+1}\} \in E$ and $f(x_\ell) \neq f(x_{\ell+1})$ as needed. □

Problem 2. Let $G = (V, E)$ be a connected graph and let S be any subset of E which does not contain a cycle. Prove that G has a spanning tree which uses every edge of S. In other words, any acyclic set of edges can be extended to a spanning tree.

Solution. Let $\mathcal{H}$ denote the set of all connected subgraphs H of G with $V(H) = V$ and $E(H) \supseteq S$. Observe that $\mathcal{H} \neq \varnothing$ since $G \in \mathcal{H}$. Fix any $H \in \mathcal{H}$ with the minimum number of edges, which is possible since G has finitely many edges. Certainly H contains every edge of S by definition; we claim that H is a spanning tree. Indeed, H is connected by definition, so we need to argue that H is acyclic.

Suppose not, so H has a cycle C. Since S is acyclic, there must be some $e \in E(C) \setminus S$. Now, $e \in E(H)$ is in a cycle and H is connected, so $H' = H - e$ is also connected. However, by construction, $H' \in \mathcal{H}$ and $|E(H')| = |E(H)| - 1$; contradicting the minimality of H. □

Problem 3. Let T, F be trees on the same vertex set. For any edge $e \in E(T) \setminus E(F)$, we know that $F + e$ contains a unique cycle: call this cycle C_e. Prove that

$$E(F) \setminus E(T) \subseteq \bigcup_{e \in E(T) \setminus E(F)} E(C_e).$$

Solution. Fix any $f \in E(F) \setminus E(T)$. Since F is a tree, we know that $F - f$ is not connected; i.e. there is some $U \subseteq V$ with $U \notin \{\varnothing, V\}$ such that f is the *only* edge of F between U and $V \setminus U$. Now,

T is connected, so there is some $e \in E(T)$ which goes between U and $V \setminus U$. Of course, $e \notin E(F)$ since $f \notin E(T)$. We claim that $f \in C_e$, which will establish the claim.

Indeed, since e goes between U and $V \setminus U$ and C_e is a cycle, there must be some $s \in E(C_e) \setminus \{e\}$ which also goes between U and $V \setminus U$ (why?). Since $E(C_e) \setminus \{e\} \subseteq E(F)$, this means that $s \in E(F)$. But then we must have $s = f$; in other words, $f \in E(C_e)$. □

Problem 4. Let $G = (V, E)$ be a weighted graph with weight function $w\colon E \to \mathbb{R}$. Suppose that G is connected and every edge has a distinct weight under w (i.e. $w(e) \neq w(s)$ for all $e \neq s \in E$). Prove that G has a *unique* minimum spanning tree.

Solution. Since G is connected, we know that G has a spanning tree; hence G must have a minimum spanning tree since G has finitely many edges. Hence, we need only focus on proving that such a spanning tree is unique.

Suppose not, then there are two distinct minimum spanning trees T_1, T_2 of G. Since $T_1 \neq T_2$, we see that $E(T_1) \triangle E(T_2) \neq \varnothing$; hence let e^* denote the edge in $E(T_1) \triangle E(T_2)$ with smallest weight. By relabeling if necessary, we may suppose that $e^* \in E(T_1)$. Consider the graph $T_2 + e^*$; since T_2 is a spanning tree and $e^* \notin E(T_2)$, $T_2 + e^*$ must contain a cycle C. Furthermore, we must have $e^* \in E(C)$. Now, since T_1 is a tree, C cannot be a subgraph of T_1 and so there is some $e^{**} \in E(C) \setminus E(T_1)$. Observe that $e^{**} \in T_2$ and that $T_2 - e^{**} + e^*$ is a spanning tree of G. Since every edge has a distinct weight and $e^{**} \in E(T_2) \setminus E(T_1) \subseteq E(T_1) \triangle E(T_2)$, we know that $w(e^*) < w(e^{**})$. But then, $T_2 - e^{**} + e^*$ is a spanning tree of G with

$$w(T_2 - e^{**} + e^*) = w(T_2) - w(e^{**}) + w(e^*) < w(T_2);$$

contradicting the fact that T_2 is a minimum spanning tree. □