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Solutions must be typeset in LaTeX. Collaboration is allowed, but must be fully acknowledged. Any collaboration should involve only discussion; all writing **must** be done individually. Use of generative AI is **forbidden**. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a set family in which $|A \cap B| < t$ for all $A \neq B \in \mathcal{F}$. Prove that

$$|\mathcal{F}| \leq \frac{\binom{n}{t}}{\binom{k}{t}}.$$

Hint: Consider sampling a uniformly random t-element subset of $[n]$.

Problem 2. A 3-term arithmetic progression (3-AP) is a set of the form $\{x, x+d, x+2d\}$ where x is an integer and d is a positive integer. Prove that there is a subset $S \subseteq [n]$ with $|S| \geq C\sqrt{n}$ which has no 3-AP. Here, $C > 0$ is some constant that is independent of n.

Problem 3. Let G be a graph. A *vertex cover* in G is a subset $A \subseteq V(G)$ such that every edge of G is incident to some vertex in A. Let $\tau(G)$ denote the size of the smallest vertex cover of G.

Suppose that G has n vertices and m edges.

1. (No probability) Prove that $\tau(G) \leq n - \alpha(G)$.

   N.b. Since $\alpha(G) \geq \frac{n}{2m}$, this implies that $\tau(G) \leq \left(1 - \frac{1}{2m}\right)n$.

2. Assuming that $m \geq n/2$, prove that $\tau(G) \leq \left(1 - \frac{n}{4m}\right)n$.