Solutions must be typeset in LaTeX. Collaboration is allowed, but must be fully acknowledged. Any collaboration should involve only discussion; all writing **must** be done individually. Use of generative AI is **forbidden**. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1. The Euler totient function of n, $\varphi(n)$, counts the number of integers in $[n]$ which are relatively prime to n. For example, if p is a prime, then $\varphi(p) = p-1$. Prove that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ (the prime factorization of n), then

$$\varphi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right).$$

Hint: WS4#3.1 may be helpful.

Problem 2. Find an expression for the number of integer solutions to $x_1 + x_2 + \cdots + x_k = n$ where $0 \leq x_i \leq i-1$ for each $i \in [k]$.

Problem 3. Use a sign-changing involution to prove the formula

$$\sum_{i=0}^{m} (-1)^i \binom{n}{i} = (-1)^m \binom{n-1}{m} \qquad \text{for } m \leq n.$$

Hint: You may find an idea in the solution WS1#4.