This document is from https://mathematicaster.org/teaching/discrete2026/hw/hw6.pdf

Solutions must be typeset in LaTeX. Collaboration is allowed, but must be fully acknowledged. Any collaboration should involve only discussion; all writing **must** be done individually. Use of generative AI is **forbidden**. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1. Let G be a connected graph a consider a function $f\colon V(G) \to X$ where X is some set. Prove that if f is *not* a constant function, then there is an edge $uv \in E(G)$ with $f(u) \neq f(v)$.

Problem 2. Let G be a graph and let A be an independent set in G. Prove that

$$\sum_{v \in A} \deg v \leq |E(G)|,$$

with equality if and only if G is bipartite with parts A and $V(G) \setminus A$.

Problem 3. Let G be a graph and fix a 2-coloring of $E(G)$. Let $R \subseteq G$ be the spanning subgraph consisting of the red edges and $B \subseteq G$ be the spanning subgraph consisting of the blue edges. Prove that $\chi(R)\chi(B) \geq \chi(G)$.

Hint: Somehow combine a proper coloring of R and of B to get a proper coloring of G.

Problem 4. The *average degree* of G is defined to be $\bar{d}(G) = \frac{1}{|V(G)|}\sum_{v \in V(G)} \deg v$ (note that $\bar{d}(G) = \frac{2|E(G)|}{|V(G)|}$ from the handshaking lemma). Prove that any graph G has a subgraph $H \subseteq G$ with $\delta(H) \geq \bar{d}(G)/2$.

Hint: Consider the smallest subgraph of G with average degree at least that of G.