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Solutions must be typeset in LaTeX. Collaboration is allowed, but must be fully acknowledged. Any collaboration should involve only discussion; all writing **must** be done individually. Use of generative AI is **forbidden**. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1. This problem is concerned with finding monochromatic trees in t-colorings of $E(K_n)$.

1. Without thinking too hard, show that any t-coloring of $E(K_n)$ has a monochromatic tree on at least $\frac{n}{t}$ many vertices.

   We will work to improve this bound.

2. Let G be a bipartite graph with parts A, B.

   (a) Prove that
   $$\sum_{\substack{a\in A, b\in B:\\ ab\in E}} (\deg a + \deg b) \geq \left(\frac{1}{|A|} + \frac{1}{|B|}\right) \cdot |E|^2.$$

   Hint: Jensen's inequality, which we will cover soon, states that $\sum_{i=1}^n a_i^2 \geq \frac{1}{n}\left(\sum_{i=1}^n a_i\right)^2$.

   (b) Prove that G contains a tree with at least $\left(\frac{1}{|A|} + \frac{1}{|B|}\right) \cdot |E|$ many vertices.
   Hint: Consider finding a "double star", which is two stars with an edge between their center vertices.

3. Show that any t-coloring of $E(K_n)$ contains a monochromatic tree on at least $\frac{n}{t-1}$ many vertices.

   Hint: Break into cases depending on whether or not the color-t-graph is connected.

4. For every n which is a multiple of 4, construct a 3-coloring of $E(K_n)$ which *does not* contain a monochromatic tree on $\frac{n}{2}+1$ many vertices. Thus, the previous part is tight when $t = 3$.

   N.b. It is known that the previous part is tight whenever an affine plane of order t exists.