This document is from https://mathematicaster.org/teaching/discrete2026/hw/hw9.pdf

Solutions must be typeset in LaTeX. Collaboration is allowed, but must be fully acknowledged. Any collaboration should involve only discussion; all writing **must** be done individually. Use of generative AI is **forbidden**. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1. Let G be a graph and $t \geq 2$ be an integer. Prove that G contains a subgraph $H \subseteq G$ (not necessarily induced) with $\chi(H) \leq t$ and $|E(H)| \geq \frac{t-1}{t}|E(G)|$.

Problem 2. Recall that we proved that every 2-coloring of $E(K_n)$ contains at least $\sim \frac{1}{4}\binom{n}{3}$ many monochromatic triangles. Show that this is optimal. That is, show that there is a 2-coloring of $E(K_n)$ with at most $\frac{1}{4}\binom{n}{3}$ many monochromatic triangles.

Problem 3. Fix integers n, s, t. Prove that if there is some $p \in [0,1]$ such that

$$\binom{n}{s} p^{\binom{s}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1,$$

then $R(s,t) > n$.

Extra fun (not required): Prove that $R(4,t) \geq C\left(\frac{t}{\log t}\right)^{3/2}$ for some constant $C > 0$. (The inequalities from WS3 will be helpful for this extra fun)

Problem 4. Fix any vectors $v_1, \ldots, v_n \in \mathbb{R}^N$, each with $\|v_i\| = 1$. Prove that there exist numbers $\epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$ such that

$$\|\epsilon_1 v_1 + \cdots + \epsilon_n v_n\| \leq \sqrt{n}.$$

Hint: Recall that $\|x\|^2 = \langle x, x \rangle$.