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Problem 1. Recall that $G_{n,p}$ is the random graph on n vertices where each edge is included independently with probability p.

- What is $\mathbb{E}|E(G_{n,p})|$?
- For $v \in V(G_{n,p})$, what is $\mathbb{E}\deg v$?
- For $u \neq v \in V(G_{n,p})$, what is $\mathbb{E}|N(u) \cap N(v)|$?

Problem 2. Consider selecting a permutation π from S_n uniformly at random and let $F(\pi)$ denote the number of fixed points of π. Determine $\mathbb{E}\,F$.

Problem 3. Let $(\Omega, \mathbf{Pr})$ be a finite probability space.

- Show that if X is a random variable on Ω, then
$$\mathbb{E}\,X = \sum_{x \in X(\Omega)} x\,\mathbf{Pr}[X = x].$$
- A pair of random variables X, Y are said to be independent if the pair of events $\{X = x\}$ and $\{Y = y\}$ are independent for all $x, y \in \mathbb{R}$. Prove that if X, Y are independent random variables on Ω, then $\mathbb{E}\,XY = \mathbb{E}\,X\,\mathbb{E}\,Y$.
- Find a pair of random variables X, Y for which $\mathbb{E}\,XY \neq \mathbb{E}\,X\,\mathbb{E}\,Y$.
- Find a pair of random variables X, Y which are *not* independent, yet $\mathbb{E}\,XY = \mathbb{E}\,X\,\mathbb{E}\,Y$.

Problem 4. Let G be any fixed graph and let $G_{1/2}$ be a random subgraph of G formed by including each edge of G independently with probability $1/2$. Prove that
$$\mathbb{E}\,\chi(G_{1/2}) \geq \sqrt{\chi(G)}.$$

N.b. It is conjectured that $\mathbb{E}\,\chi(G_{1/2}) \geq C\frac{\chi(G)}{\log \chi(G)}$ for some constant $C > 0$, which would be best-possible if true.

Hint: The AM–GM inequality ($\sqrt{ab} \leq \frac{a+b}{2}$) along with HW6#3 may be helpful.