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Problem 1 (Even vs odd, revisited). Use the binomial theorem to prove that, for $n \geq 1$, the number of even-sized subsets of $[n]$ is the same as the number of odd-sized subsets of $[n]$.

Problem 2 (Committee selection). Find a combinatorial proof that

$$\binom{n}{r} 2^{n-r} = \sum_{k=r}^n \binom{n}{k} \binom{k}{r}$$

Problem 3 (Stirling numbers). The Stirling numbers (of the second kind) $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ count the total number of ways to partition an n -element set into k non-empty pieces.

- Prove the identity

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\} + k \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\}.$$

- For a number x and a non-negative integer n , the falling-factorial is $x^{\underline{n}} = x(x-1) \cdots (x-n+1)$ with the convention that $x^{\underline{0}} = 1$ always. Prove the following binomial-theorem-esque identity:

$$x^n = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} x^{\underline{k}}.$$

Problem 4. Determine, with proof, the total number of pairs (A, B) where $A \subseteq B \subseteq [n]$. For extra fun, find two different proofs: one using the binomial theorem and one using a direct bijection.