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Problem 1. Fix $n \geq 2$. Prove that there is a matrix in $\{\pm 1\}^{n \times n}$ such that all row/column sums are bounded above by $C\sqrt{n \log n}$ in absolute value. Here $C > 0$ is some constant.

Problem 2. Fix $\epsilon > 0$ and a graph G on n vertices. We say that a 2-coloring of $V(G)$ is ϵ-balanced if

$$|\deg_r v - \deg_b v| \leq \epsilon \deg v \qquad \text{for all } v \in V.$$

Prove that there is a constant C such that if $\delta(G) \geq C\frac{\log n}{\epsilon^2}$, then G has a ϵ-balanced 2-coloring.

Problem 3. Improve Chernoff's bound for sums of independent Bernoulli random variables with small means.

- Suppose that X is a random variable satisfying $0 \leq X \leq 1$. Prove that $\mathbb{E}\, e^{tX} \leq e^{(e^t-1)\,\mathbb{E}\, X}$ for every $t \in \mathbb{R}$.

- Suppose that $X_1, \ldots, X_n$ are independent random variables satisfying $0 \leq X_i \leq 1$ for every i. Set $S = X_1 + \cdots + X_n$ and $\mu = \mathbb{E}\, S$. Prove that

$$\mathbf{Pr}[S \geq (1+\delta)\mu] \leq \exp\left(-\frac{\delta^2}{2+\delta}\mu\right) \qquad \text{for all } \delta > 0,$$
$$\mathbf{Pr}[S \leq (1-\delta)\mu] \leq \exp\left(-\frac{\delta^2}{2}\mu\right) \qquad \text{for all } 0 < \delta < 1.$$

  Hint: The following inequalities may be helpful:

$$\log(1+x) \geq \frac{2x}{2+x} \quad \text{for all } x > 0, \qquad \text{and} \qquad \log(1-x) \geq -\frac{x(2-x)}{2(1-x)} \quad \text{for all } 0 < x < 1.$$

- Consider $G_{n,p}$ where $p = \frac{c}{n-1}$ for some fixed $c > 0$. Prove that there is a constant C (which may depend on c) such that

$$\mathbf{Pr}[\Delta(G_{n,p}) \leq C \log n] \to 1 \qquad \text{as } n \to \infty.$$

  Can the Chernoff bound proved in class be used to establish this fact?