These notes are from https://mathematicaster.org/teaching/graphs2022/extra_02-08.pdf

The proof of Theorem 4.2 in the book skipped important details, and these details are not trivial, so here is a full proof.

Theorem 1. *Let G be a graph. If there are $x, y \in V(G)$ for which there are at least two x-y paths in G, then G contains a cycle.*

We give two proofs.

Proof #1. Notice that there is always exactly one x-x path for any vertex x, and so we must have $x \neq y$. Call the two of the x-y paths $(x = u_0, u_1, \dots, u_k = y)$ and $(x = v_0, v_1, \dots, v_\ell = y)$; note that $k, \ell \geq 1$ since $x \neq y$.

Let i be the largest index for which $u_j = v_j$ for all $j \in \{0, \dots, i\}$. Note that i exists since $u_0 = x = v_0$. Additionally, we see that $i < \min\{k, \ell\}$. Indeed, suppose, without loss of generality, that $k \leq \ell$ and that $i = k$. If $k = \ell$, then this means t hat $u_j = v_j$ for all j and so the two paths are the same, which we know is note the case. If $k < \ell$, then $y = u_k = v_k$, but this is impossible since $v_k \neq v_\ell = y$.

Therefore, u_{i+1} and v_{i+1} exist and $u_{i+1} \neq v_{i+1}$. Now, let $s \in \{i+1, \dots, k\}$ be the smallest index for which $u_s \in \{v_{i+1}, \dots, v_\ell\}$. Note that s exists since $u_k = y = v_\ell$. Thus, suppose that $t \in \{i+1, \dots, \ell\}$ is such that $u_s = v_t$. We know that either $s \neq i+1$ or $t \neq i+1$ since $u_{i+1} \neq v_{i+1}$ from above.

We therefore see that $(v_i = u_i, u_{i+1}, \dots, u_s = v_t, v_{t-1}, \dots, v_{i+1})$ forms a cycle in G as needed. ☐

Proof #2. Among all pairs of vertices with at least two paths between them, let x and y be a pair with $d(x, y)$ minimum. Of course, $d(x, y) \geq 1$ since there is exactly one path from a vertex to itself. Set $d = d(x, y)$ and let $(x = u_0, u_1, \dots, u_d = y)$ be any x-y geodesic. Since there are at least two x-y paths, we can find a different path, call it $(x = v_0, v_1, \dots, v_k = y)$. We claim that $\{u_1, \dots, u_{d-1}\} \cap \{v_1, \dots, v_{k-1}\} = \varnothing$. Suppose for the sake of contradiction that these sets intersect, and so $u_s = v_t$ for some $s \in [d-1]$ and $t \in [k-1]$. Since $(x = u_0, \dots, u_d = y)$ is a geodesic, note that $d(x, u_s) = s$ and $d(u_s, y) = d - s$. Now, $W_1 = (x = u_0, \dots, u_s)$ and $W_2 = (x = v_0, \dots, v_t = u_s)$ are two x-u_s paths and $W_3 = (u_s, u_{s+1}, \dots, u_d = y)$ and $W_4 = (u_s = v_t, v_{t+1}, \dots, v_k = y)$ are two u_s-y paths. While it could be the case that $W_1 = W_2$ or that $W_3 = W_4$, it cannot be the case that both of these equalities hold since the original paths were distinct.

Case 1: $W_1 \neq W_2$. Then W_1 and W_2 are two different x-u_s paths; a contradiction to the minimality of x, y since $d(x, u_s) = s < d$.

Case 2: $W_3 \neq W_4$. Then W_3 and W_4 are two different u_s-y paths; a contradiction to the minimality of x, y since $d(u_s, y) = d - s < d$.

Therefore, $\{u_1, \dots, u_{d-1}\} \cap \{v_1, \dots, v_{k-1}\} = \varnothing$ and so $(x = u_0, u_1, \dots, u_d = y = v_k, v_{k-1}, \dots, v_1)$ forms a cycle in G. ☐