

This homework is from <https://mathematicaster.org/teaching/graphs2022/hw11.pdf>

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (2pts). Let G be any graph. Prove that

$$|E(G)| \geq \binom{\chi(G)}{2}.$$

Problem 2 (1pt). Generalize Theorem 1 from 04-07:

Let $G = (V, E)$ be any graph and consider coloring the edges of G red and blue (formally, we have a function $f: E \rightarrow \{\text{red, blue}\}$). Let G_r be the graph formed by the red edges and let G_b be the graph formed by the blue edges (formally, $G_r = (V, f^{-1}(\text{red}))$ and $G_b = (V, f^{-1}(\text{blue}))$). Prove that

$$\chi(G_r) \cdot \chi(G_b) \geq \chi(G).$$

Problem 3 (1pt). For every pair of positive integers m, n , construct a graph G with the following properties:

- G has $m \cdot n$ many vertices, and
- $\chi(G) = m$, and
- $\chi(\overline{G}) = n$.

Problem 4 (2pts). Prove that G is 3-critical if and only if $G \cong C_{2n+1}$ for some positive integer n .

Problem 5 (2pts). Fix any integer $k \geq 1$. Prove that if G is an n -vertex graph wherein

$$\chi(G[N(v)]) \leq k, \quad \text{for every } v \in V(G),$$

then $\chi(G) \leq \sqrt{2kn}$.

(Hint: Take motivation from our proof that $\chi(G) \leq \sqrt{2n}$ if G is triangle-free. Be warned, though, there are a couple steps which require more care.)

(Hint: If $0 \leq x \leq y$, then $x \leq \sqrt{xy}$.)

Problem 6 (2pts). For graphs G, H , the graph G is said to be H -free if G does not contain a copy of H .

Fix any integer $k \geq 3$. Prove that if G is a C_k -free graph on n vertices, then $\chi(G) \leq \sqrt{2(k-2)n}$.
(Hint: Problem 5)