This homework is from https://mathematicaster.org/teaching/graphs2022/hw2.pdf

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (1 pt). Recall that a *directed graph* (or digraph) is a pair $D = (V, E)$ where V is a set and $E \subseteq V^2$. For $u, v \in V$, a u-v diwalk in D is a sequence $(u = v_0, \ldots, v_k = v)$ such that $(v_i, v_{i+1}) \in E$ for all $i \in \{0, \ldots, k-1\}$.

   Consider the relation R on V where $u\,R\,v$ iff there is a u-v diwalk. Give an example of a digraph D where R is **not** an equivalence relation. Justify your answer. (Feel free to draw a picture of D)

Problem 2 (2 pts). Let G be a connected graph and consider a function $f \colon V(G) \to X$ where X is some arbitrary set. Prove that if f is *not* a constant function, then there is an edge $uv \in E(G)$ such that $f(u) \neq f(v)$.

Problem 3 (2 pts). Prove that every graph on at least two vertices has a pair of vertices with the same degree.

Problem 4 (2 pts). Prove that if G is a graph with $\delta(G) \geq 2$, then G must contain a cycle.

Problem 5 (3 pts). Let G be a graph and let A be an independent set of G. Prove that

$$\sum_{v \in A} \deg v \leq |E(G)|$$

with equality if and only if G is bipartite with parts A and $V(G) \setminus A$.
(Especially in this problem, be sure to carefully justify all steps in your argument)