**MATH 314** | **Discussion Session #1 Solutions** | **Feb 3**

These solutions are from https://mathematicaster.org/teaching/graphs2022/sol-ds1.pdf

I encourage you to first read through all of these problems and then focus first on those with which you're less comfortable.

**Problem 1.** State three equivalent definitions of graph connectivity.

**Solution.** Walks & paths & breaks, oh my! $\square$

**Problem 2.** Show that any minim**al** $u$-$v$ walk is a $u$-$v$ path.

**Solution.** Consult your notes. $\square$

**Problem 3.** Prove that for any $u, v, w \in V(G)$, $d(u, w) \leq d(u, v) + d(v, w)$. (Pay careful attention if $G$ happens to be disconnected!)

**Solution.** If $u, v, w$ are in the same connected component of $G$, then this isn't two bad: just use the definition of distance and build a $u$-$w$ path which passes through $v$. If some of $u, v, w$ are in different connected components, then this is a case-check to ensure you never wind up with $\infty \leq$ something finite. $\square$

**Problem 4.** Let $G$ be a graph. Define the relation $R$ on $V(G)$ where $u\,R\,v$ if and only if for every partition $V(G) = A \sqcup B$ with $u \in A$ and $v \in B$, there is at least one edge between $A$ and $B$. Prove that $R$ is an equivalence relation on $V(G)$.

**Solution.** Crank through the three properties of an equivalence relation. The only one which is somewhat interesting is transitivity. $\square$

**Problem 5.** Let $G$ be a connected graph Suppose that there are no three distinct vertices $u, v, w \in V(G)$ such that $uv, uw \in E(G)$ yet $vw \notin E(G)$. Prove that if $G$ has $n$ vertices, then $G \cong K_n$.

**Solution.** [#1] Let $R$ be the relation on $V(G)$ where $u\,R\,v$ iff either $u = v$ or $uv \in E(G)$.

1. Prove that $R$ is an equivalence relation.
2. What do the equivalence classes of $R$ look like?
3. Prove that $R$ has only one equivalence class.

$\square$

**Solution.** [#2] Prove this by induction on $n$ using the fact that if $G$ is connected and has at least two vertices, then $G - v$ is connected for some $v \in V(G)$. $\square$

**Problem 6.** Prove the "handshaking dilemma": If $D = (V, E)$ is a digraph, then

$$\sum_{v \in V} \deg^+ v = \sum_{v \in V} \deg^- v = |E|.$$

Here, $\deg^+ v = |\{u \in V : (v, u) \in E\}|$ and $\deg^- v = |\{u \in V : (u, v) \in E\}|$ denote the out- and in-degree of $v$, respectively.

**Solution.** Simply use the fact that if $X, Y$ are finite sets and $\Omega \subseteq X \times Y$, then

$$|\Omega| = \sum_{x \in X} |\{y \in Y : (x,y) \in \Omega\}| = \sum_{y \in Y} |\{x \in X : (x,y) \in \Omega\}|.$$

$\square$

**Problem 7.** Fix a graph $G = (V, E)$. The *line graph* of $G$, denoted $L(G)$, has vertex set $E$ and edge set $\left\{es \in \binom{E}{2} : e \cap s \neq \varnothing\right\}$.

1. Prove that $L(C_n) \cong C_n$ for every $n \geq 3$.

2. Prove that $L(P_n) \cong P_{n-1}$ for every $n \geq 2$.

3. For $uv \in E$, prove that $\deg_{L(G)}(uv) = \deg_G(u) + \deg_G(v) - 2$.

4. Prove that $L(G)$ is connected whenever $G$ is connected.

5. If $L(G)$ is connected, must it be the case that $G$ is connected?

**Solution.**

1. If we label $V(C_n) = \{0, 1, \ldots, n-1\}$ and $E(C_n) = \big\{\{i, i+1 \pmod{n}\} : i \in V(C_n)\big\}$, then one possible isomorphism would be $f\colon L(C_n) \to C_n$ defined by $f\big(\{i, i+1 \pmod{n}\}\big) = i$.

2. If we label $V(P_n) = [n]$ and $E(P_n) = \big\{\{i, i+1\} : i \in [n-1]\big\}$, then one possible isomorphism would be $f\colon L(P_n) \to P_{n-1}$ defined by $f\big(\{i, i+1\}\big) = i$.

3. Which edges touch the edge $uv$?

4. Think about the actual edges you traverse in a walk in $G$.

5. Nope, what if $G$ has isolated vertices? Although, if $G$ has no isolated vertices, then the statement is true.

$\square$

**Problem 8.** Consider a digraph $D = (V, E)$. Recall that a $u$-$v$ diwalk is a sequence of vertices $(u = v_0, \ldots, v_k = v)$ such that $(v_i, v_{i+1}) \in E$ for all $i \in \{0, \ldots, k-1\}$. $D$ is said to be *strongly connected* if for every $u, v \in V$, there is both a $u$-$v$ diwalk and a $v$-$u$ diwalk (consult HW2.1 for why this definition is reasonable).

Prove that $D$ is strongly connected if and only if for any partition $V = A \sqcup B$ with $A, B$ nonempty, there is $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that $(a_1, b_1), (b_2, a_2) \in E$ (note that we could have $a_1 = a_2$ and/or $b_1 = b_2$).

**Solution.** This is basically identical to what we proved in class with breaks in undirected graphs — you just have to do every step twice. $\square$

**Problem 9.** Show that there is no graph $G$ with the property that $\deg_G v \equiv \deg_{\overline{G}} v \equiv 1 \pmod{2}$ for every $v \in V(G)$.

**Solution.** How many vertices must such a $G$ have? Is this a problem? $\square$

**Problem 10.** Prove that $G \cong H$ if and only if $\overline{G} \cong \overline{H}$.

**Solution.** This basically falls right out of the definition of an isomorphism. □

**Problem 11.** Let $G$ be a bipartite graph with parts $A, B$. Show that

$$\sum_{a \in A} \deg a = \sum_{b \in B} \deg b.$$

**Solution.** We could just invoke HW2.5, but we can get a more direct proof by considering the set $\{(a, b) \in A \times B : ab \in E(G)\}$. □

**Problem 12.** A *set system* is a pair $(V, E)$ where $V$ is a set and $E \subseteq 2^V$ (i.e. a collection of subsets of $V$). (Note that a graph is simply a set system wherein each set has size 2). For a set-system $(V, E)$, we can define the degree of a vertex $v \in V$ to be $\deg v = |\{e \in E : v \in e\}|$, exactly the same as we did with graphs.

Use problem 11 to prove the generalization of the handshaking lemma to a set system $(V, E)$:

$$\sum_{v \in V} \deg v = \sum_{e \in E} |e|.$$

(Observe how this implies the original handshaking lemma).

**Solution.** Let $G = (V, E)$ be a set system; we build a (normal) bipartite graph $H$ as follows: $V(H) = V \sqcup E$ where, for $v \in V$ and $e \in E$, $ve \in E(H)$ if and only if $v \in e$.

Now, observe that for any $v \in V$, $\deg_H v = \deg_G v$ and for any $e \in E$, $\deg_H e = |e|$. Then apply problem 11. □

**Problem 13.** A group of sociologists are curious about the affect of eye color on childhood friendship. So they gather a group of 100 children, each having either brown or blue eyes, and observe the children's play over the course of a day. After gathering the data, the researchers are surprised to find that the average blue-eyed child played with 3 times as many brown-eyed children as the number of blue-eyed children the average brown-eyed child played with. Based on this disparity, the sociologists conclude that blue-eyed children are significantly more likely to look past physical differences and make friends.

Later, you (a student in this graph theory class) have dinner with your friend who was part of this group of sociologists, and they explain the experiment and their excitement about the results. In confusion, you reply, "Why are you so excited? Did you not know there were 75 brown-eyed children and 25 blue-eyed children in your study?"

Explain your response.

**Solution.** Compare with problem 11.

N.b. I have seen basically this exact issue pop up in real "studies", so be careful to think critically about the statistics you may encounter in your day-to-day life. □