These solutions are from https://mathematicaster.org/teaching/graphs2022/sol-ds2.pdf

I encourage you to first read through all of these problems and then focus first on those with which you're less comfortable.

Problem 1. Show that every tree is bipartite.

Solution. Trees don't have *any* cycles... □

Problem 2. Show that if G has $|E(G)| = |V(G)| + k$ for some integer $k \geq -1$, then G contains at least $k+1$ distinct cycles (though these cycles may overlap substantially).

Solution. Prove it by induction on k. The case of $k = -1$ is trivial (every graph has at least 0 cycles). For $k \geq 0$, delete any edge e from G which lives in a cycle (why must one exist?); then $G - e$ has one fewer edge and all cycles in G which used the edge e no longer exist. □

Problem 3 (Fulkerson–Hoffman–McAndrew conditions)**.** Prove that if $d_1 \geq \cdots \geq d_n$ is graphical, then $\sum_{i=1}^n d_i$ is even and

$$\sum_{i=1}^{k} d_i \leq k(m-1) + \sum_{i=m+1}^{n} \min\{d_i, k\}, \qquad \text{for every } k, m \in [n] \text{ with } k \leq m.$$

Note that these conditions imply the Erdős–Gallai conditions (why?) and so this is actually a biconditional statement.

Solution. Use the set

$$\Omega = \big\{(x, y) \in \{v_1, \dots, v_k\} \times V : xy \in E\big\}.$$

The LHS is $|\Omega|$ as is seen by summing over the first coordiante. For the RHS, partition $\Omega = \Omega_1 \sqcup \Omega_2$ where

$$\Omega_1 = \big\{(x, y) \in \{v_1, \dots, v_k\} \times \{v_1, \dots, v_m\} : xy \in E\big\},$$
$$\Omega_2 = \big\{(x, y) \in \{v_1, \dots, v_k\} \times \{v_{m+1}, \dots, v_n\} : xy \in E\big\},$$

and upper bound the size of each set. □

Problem 4 (Bollobás conditions)**.** Prove that if $d_1 \geq \cdots \geq d_n$ is graphical, then $\sum_{i=1}^n d_i$ is even and

$$\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} \min\{d_i, k-1\} + \sum_{i=k+1}^{n} \min\{d_i, k\}, \qquad \text{for every } k \in [n].$$

Note that these conditions imply the Erdős–Gallai conditions (why?) and so this is actually a biconditional statement.

Solution. Use the set
$$\Omega = \big\{(x, y) \in \{v_1, \dots, v_k\} \times V : xy \in E\big\}.$$
The LHS is $|\Omega|$ as is seen by summing over the first coordinate. For the RHS, upper bound $|\Omega|$ by summing over the second coordinate. □

Problem 5 (Grünbaum conditions)**.** Prove that if $d_1 \geq \cdots \geq d_n$ is graphical, then $\sum_{i=1}^n d_i$ is even and
$$\sum_{i=1}^{k} \max\{d_i, k-1\} \leq k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\}, \qquad \text{for every } k \in [n].$$
Note that these conditions imply the Erdős–Gallai conditions (why?) and so this is actually a biconditional statement.

Solution. Use the set
$$\Omega = \big\{(x, y) \in \{v_1, \dots, v_k\} \times V : xy \in E \ \vee \ y \in \{v_1, \dots, v_k\} \setminus \{x\}\big\}.$$
For the LHS, notice that for any fixed $x \in \{v_1, \dots, v_k\}$,
$$\big\{y \in V : (x, y) \in \Omega\big\} \supseteq \big(\{v_1, \dots, v_k\} \setminus \{x\}\big) \cup N(x),$$
which has size at least $\max\{k-1, d_i\}$. The RHS is essentially the same as our proof of the necessity of the Erdős–Gallai conditions done in class. □

Problem 6 (Ryser conditions)**.** A pair of sequences $(a_1, \dots, a_m), (b_1, \dots, b_n)$ is said to be *bipartite-graphical* if there is a bipartite graph G with parts $A = \{v_1, \dots, v_m\}$ and $B = \{u_1, \dots, u_n\}$ such that $\deg v_i = a_i$ for all $i \in [m]$ and $\deg u_i = b_i$ for all $i \in [n]$.

Suppose that $d_1, \dots, d_n$ is a graphical sequence. Show that if $\gamma_1, \dots, \gamma_n$ is any sequence such that $\gamma_i \in \{d_i, d_i + 1\}$ for all $i \in [n]$, then the pair of sequences $(\gamma_1, \dots, \gamma_n), (\gamma_1, \dots, \gamma_n)$ is bipartite-graphical.

Turns out that this is actually a biconditional statement if we include the condition that $\sum_{i=1}^n d_i$ is even. That is to say that if $\sum_{i=1}^n d_i$ is even and the pair $(\gamma_1, \dots, \gamma_n), (\gamma_1, \dots, \gamma_n)$ is bipartite-graphical for all sequences $\gamma_1, \dots, \gamma_n$ with $\gamma_i \in \{d_i, d_i + 1\}$ for all i, then $d_1, \dots, d_n$ is graphical. (I don't expect you to prove this.)

Solution. Let A and B be two copies of the vertex-set of G (where G is a realization of $d_1, \dots, d_n$) and connect vertices between the two parts that correspond to edges in G. Then add the edge between the two copies of an individual vertex if $\gamma_i = d_i + 1$. □

Problem 7. This problem will walk through another proof that trees on n vertices have $n - 1$ edges. Let T be a tree and fix any vertex $u \in V(T)$. For each non-negative integer i, set $N_i = \{v \in V(T) : d(u, v) = i\}$. Note that $N_0 = \{u\}$ and that N_1 is the neighborhood of u.

1. Show that $V(T) = \bigsqcup_{i \geq 0} N_i$.
2. Show that if $xy \in E(T)$, then there is some $i \geq 0$ such that $x \in N_i$ and $y \in N_{i+1}$ (or vice versa).
3. Show that for each $i \geq 1$ and any $v \in N_i$, v has exactly one neighbor in N_{i-1}.
4. Use these facts to prove that $|E(T)| = |V(T)| - 1$.

Solution.

1. T is connected so each vertex is at some non-negative, integer distance from u, and these sets are disjoint since $d(u, v)$ is a fixed number.

2. Show first that if $x \in N_i$ and $y \in N_j$ then $|j - i| \leq 1$. Then show that it's impossible that $i = j$.

3. We know that there's at least one neighbor there; why can't there be two?

4. Use the bipartite handshaking lemma (HW2.5) to count the edges between N_i and N_{i+1} for all $i \geq 0$. Then add them all up!

$\square$

Problem 8. This problem shows that the main idea in problem 7 actually classifies all trees. Let G be a graph and suppose that there is a partition $V(G) = V_0 \sqcup \cdots \sqcup V_k$ with the following properties:

1. V_i is non-empty for all $i \in \{0, \ldots, k\}$, and

2. $G[V_0]$ is a tree, and

3. V_i is an independent set for all $i \in [k]$, and

4. For each $i \in [k]$ and any $v \in V_i$, v has exactly one neighbor in $\bigcup_{j=0}^{i-1} V_j$.

Show that G is a tree.

Solution. First, show by induction on $i \in \{0, \ldots, k\}$ that $G[V_0 \cup \cdots \cup V_i]$ is a connected graph. Then to show that G is acyclic, suppose there is a cycle C and look at the largest i for which $V(C) \cap V_i \neq \varnothing$ — does this cause a problem? $\square$

Problem 9 (Borůvka's algorithm)**.** Let G be a connected graph and let $w \colon E(G) \to \mathbb{R}$ be a weight function which assigns distinct weights (i.e. $w(e) \neq w(s)$ for any distinct $e, s \in E(G)$).

Initialize $F = (V(G), \varnothing)$ and iterate the following process:

1. If F is connected, terminate and return F.

2. If F is disconnected, suppose that $F_1, \ldots, F_k$ are the connected components of F. For each $i \in [k]$, let e_i be the minimum weight edge which has exactly one vertex in F_i (note: there cannot be any ties since w assigns distinct weights). Replace F by $F + e_1 + \cdots + e_k$ and repeat (note that it's possible that some of the e_i's are the same — if this happens, we add that edge in only once).

Prove that this algorithm returns a minimum weight spanning tree of G. (Note: The most difficult part is arguing that the graph returned is acyclic.)

Extra fun: Why did we need to require that w assigned distinct weights? How could you modify Borůvka's algorithm if this is not the case?

Extra extra fun: Show that Borůvka's algorithm terminates after at most $\lfloor \log_2 |V(G)| \rfloor$ iterations.

Solution. The proof is pretty similar to Kruskal and Prim at its heart. However, there's one crucial difference. In Kruskal and Prim, we added only a single edge per iteration, which made it easy to verify that the partial tree was acyclic at each step. However, here we are adding multiple edges at a time (and these edges are chosen without any knowledge of one another)... What prevents this from creating a cycle? The fact that w assigns distinct weights is crucial to this argument.

Once you know that Borůvka actually returns a spanning tree, the argument that it's a minimum weight spanning tree is very similar to the argument used in Kruskal and Prim. □

Problem 10 (Challenge question ☺)**.** Show that if $|E(G)| \geq 2|V(G)|$, then G contains a cycle of length at most $2\lceil \log_2 |V(G)| \rceil - 1$.

Solution. First show that we may suppose that G is connected. Then proceed by induction on $n = |V(G)|$ (base case of $n = 5$ makes sense here) and argue by contradiction.

Fix your favorite vertex $u \in V(G)$ and for each non-negative integer i, set $N_i = \{v \in V(G) : d(u,v) = i\}$. Set $m = \lceil \log_2 n \rceil - 1$ and prove that $G[N_0 \cup \cdots \cup N_m]$ is a tree.

Next consider any $\ell \in [m]$ and suppose that $|N_\ell| \leq \sum_{i=0}^{\ell-1} |N_i|$. Setting $A = \bigcup_{i=0}^{\ell-1} N_i$, show that A is incident to at most $2|A| - 1$ many edges. Thus, we may remove the vertices in A from G to form a new graph G' which has $n - |A|$ many vertices and at least $2|E(G)| - 2|A| + 1 \geq 2(n - |A|)$ edges; so we may apply induction and win since $|A| \geq 1$.

Therefore, we know that $|N_\ell| \geq 1 + \sum_{i=0}^{\ell-1} |N_i|$ for every $\ell \in [m]$. Prove by induction on ℓ that this implies that $|N_\ell| \geq 2^\ell$ for each $\ell \in [m]$. Set $a = |\bigcup_{\ell=0}^{m} N_\ell|$; show that $a \geq 2^{m+1} - 1$.

Now, set $b = |\bigcup_{i \geq m+1} N_i|$; note that $n = a + b$. Argue that $|E(G)| \leq (a-1) + ab + \binom{b}{2}$ and use this to show that $b \geq 2$.

Now the kicker:

$$n = a + b \geq (2^{m+1} - 1) + 2 = 2^{m+1} + 1 = 2^{\lceil \log_2 n \rceil} + 1 \geq n + 1;$$

contradiction! □