These solutions are from https://mathematicaster.org/teaching/graphs2022/sol-ds3.pdf

I encourage you to first read through all of these problems and then focus first on those with which you're less comfortable.

Problem 1. Prove that if $\Delta(G) \le 2$, then $\kappa(G) = \lambda(G)$.

Solution. Since $\kappa(G) = \lambda(G) = 0$ if G is disconnected, we may suppose throughout that G is connected.

The cases when $\Delta(G) \in \{0, 1\}$ will be immediate (the only options are K_1 and K_2, respectively). Prove that if $\Delta(G) = 2$, then G is either a path or a cycle. Then argue that $\kappa(G) = \lambda(G)$ for these graphs. □

Problem 2. Recall that $\kappa(G) \le \lambda(G) \le \delta(G)$ for any graph G. Convince yourself that these inequalities can be strict. Furthermore, show that for any integers κ, λ, δ with $1 \le \kappa < \lambda < \delta$, there is a graph G with $\kappa(G) = \kappa$, $\lambda(G) = \lambda$ and $\delta(G) = \delta$.

Solution. There are many constructions, but here is just one.

Let $U = \{u_1, \dots, u_{\delta+1}\}$ and $U' = \{u'_1, \dots, u'_{\delta+1}\}$ and form a graph G with vertex-set $U \sqcup U'$ as follows:

- Include all edges among U and all edges among U', so $G[U] \cong G[U'] \cong K_{\delta+1}$.
- For all $i \in [\lambda - \kappa + 1]$, include the edge $u_1 u'_i$.
- For all $i \in \{2, \dots, \kappa\}$, include the edge $u_i u'_i$.

Show that $\kappa(G) = \kappa$, $\lambda(G) = \lambda$ and $\delta(G) = \delta$ for this G. □

Problem 3. Let G be a graph and let $A, B \subseteq V(G)$ be non-empty subsets (that could intersect). An A-B path is a path $(v_0, \dots, v_k)$ with $v_0 \in A$, $v_k \in B$ and none of $v_1, \dots, v_{k-1}$ are in either A or B (we used these paths in our proof of Menger's theorem). Note that (x) is an A-B path if and only if $x \in A \cap B$.

1. Use Menger's theorem for vertex-connectivity to prove that if $|A|, |B| \ge \kappa(G)$, then there are at least $\kappa(G)$ many vertex-disjoint A-B paths in G.

   N.b. This fact also follows immediately from Lemma 1 in the extra notes from 03-03; one just needs to observe that $\kappa_G(A, B) \ge \kappa(G)$ in this case.

2. Use Menger's theorem for edge-connectivity to prove that if $|A|, |B| \ge \lambda(G)$, then there are at least $\lambda(G)$ many edge-disjoint A-B paths in G such that each vertex in $A \cup B$ belongs to at most one of these paths (i.e. no two start at nor end at the same point).

Solution. Augment G by adding two vertices a, b where a is adjacent to everything in A and b is adjacent to everything in B; call this new graph G'. Show that any a-b path in G' contains an A-B path in G. Then show that $\kappa(G') \ge \kappa(G)$ and $\lambda(G') \ge \lambda(G)$. Finally, apply Menger's theorem (either vertex- or edge-version) to get the desired a-b paths in G' and thus the desired A-B paths in G. □

Problem 4. Suppose that a graph G has blocks $B_1, \dots, B_k$ and cut-vertices $v_1, \dots, v_\ell$. We build a new graph $\mathcal{B}$ with $V(\mathcal{B}) = \{b_1, \dots, b_k, c_1, \dots, c_\ell\}$ and $c_i b_j \in E(\mathcal{B})$ if and only if $v_i \in V(B_j)$. Note that $\mathcal{B}$ is a bipartite graph with parts $\{b_1, \dots, b_k\}$ and $\{c_1, \dots, c_\ell\}$.

Prove that if G is a connected graph on at least two vertices, then $\mathcal{B}$ is a tree. Furthermore, show that none of $c_1, \dots, c_\ell$ are leaves of $\mathcal{B}$. $\mathcal{B}$ is sometimes called the *block tree* of G.

Solution. Consider first the case that G has only a single block (i.e. $k = 1$).

Next, show that for each $i \in [\ell]$, c_i is adjacent to at least two b_j's (thus, once we show that $\mathcal{B}$ is a tree, we will know that none of the c_i's are leaves).

In order to show that $\mathcal{B}$ is connected, it is enough to show that there is a walk between any pair of b_i's (since each c_j is connected to at least one of these). Do so by turning a path in G between a vertex in B_i and a vertex in B_j into a walk in $\mathcal{B}$ between b_i and b_j.

In order to show that $\mathcal{B}$ is acyclic, proceed by contradiction. Take the *shortest* cycle in $\mathcal{B}$ and show how this cycle implies the existence of a cycle in G which uses edges from different blocks, thus reaching a contradiction. □

Problem 5. The *degeneracy* of a graph G is defined to be

$$d(G) = \max\big\{\delta(H) : H \text{ is a subgraph of } G\big\}.$$

1. Prove that $d(G) \leq 1$ if and only if G is a forest.

2. Prove that $d(G)$ is the smallest integer d such that there is an ordering $V(G) = \{v_1, \dots, v_n\}$ so that $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| \leq d$ for all $i \in [n]$.

   Such orderings will be our friend once we start talking about coloring graphs :)

3. Prove that if G is connected and $d(G) = \Delta(G)$, then G is regular.

The *arboricity* of a graph G, denoted by $a(G)$, is defined to be the minimum integer k such that we can partition the edges of G into k forests. If G has no edges, we set $a(G) = 0$.

4. Prove that $d(G) \geq a(G)$.

5. Prove that $d(G) \leq 2a(G) - 1$.

Solution.

1. ($\Leftarrow$) Every subgraph of a forest is also a forest.

   ($\Rightarrow$) Cycles have min-degree 2.

2. First find a labeling with $d = d(G)$. Do so by induction on n, noting that if H is a subgraph of G, then $d(H) \leq d(G)$. To accomplish the induction step, consider letting v_n be any smallest-degree vertex of G.

   To finish the problem, we must show that it is impossible to have $d < d(G)$. Fix an ordering with $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| \leq d$ for all $i \in [n]$. To do so, for any subgraph H of G, consider the smallest index $i \in [n]$ such that H is a subgraph of $G[\{v_1, \dots, v_i\}]$ and use this to show that $\delta(H) \leq d$. Observe how this implies that $d \geq d(G)$.

3. Prove that if H is a proper subgraph of G (i.e. $H \neq G$), then there is some $v \in V(H)$ with $\deg_H v < \deg_G v$. To accomplish this, consider separately the cases when H is spanning and when H is not spanning (the connectivity of G will be essential for the latter case). Conclude that $\delta(H) < \Delta(G)$ for all proper subgraphs of G. Thus, if $d(G) = \Delta(G)$, then $d(G) = \delta(G)$ and so G is regular.

4. Consider an ordering satisfying part 2 with $d = d(G)$. Build a forest partition $F_1, \ldots, F_d$ of G by, for each $i \in [n]$, distributing the $\leq d$ edges incident to v_i in $G[\{v_1, \ldots, v_i\}]$ into the F_i's so that each one of these $\leq d$ edges gets assigned to a different F_i. By construction, the same ordering implies that $d(F_i) \leq 1$ and so it is a forest. Since we have covered each edge, these forests show that $a(G) \leq d = d(G)$.

5. Observe that if H is a subgraph of G, then $a(H) \leq a(G)$. Now, prove that $|E(G)| \leq a(G) \cdot (|V(G)| - 1)$ (how many edges does a forest have?). Then invoke the handshaking lemma to find that $\delta(G) \leq 2a(G) - 1$.

□

Problem 6. Let G be a graph and let $\mathcal{I}$ be any collection of independent sets of G. For a vertex $v \in V(G)$, let $\mathcal{I}_v = \{I \in \mathcal{I} : v \in I\}$ be the set of those independent sets in $\mathcal{I}$ which contain the vertex v. Say that $v \in V(G)$ is *uncommon* if $|\mathcal{I}_v| \leq |\mathcal{I}|/2$ and otherwise say that v is *common*.

1. Prove that if G has at least one edge, then G has an uncommon vertex.

2. Prove that if G is *not* bipartite, then there is an edge $uv \in E(G)$ such that both u and v are uncommon.

3. Open question: Suppose that $\mathcal{I}$ is the set of all maxim**al** independent sets of G. If G has at least one edge, then there is some $uv \in E(G)$ such that both u and v are uncommon.

Solution.

1. Prove that if $uv \in E(G)$, then $\mathcal{I}_u$ and $\mathcal{I}_v$ are disjoint, which implies that at least one of u, v is uncommon.

2. Let A be the set of uncommon vertices and let B be the set of common vertices. Certainly $V(G) = A \sqcup B$. The suggested proof of part 1 implies that B is an independent set. Since G isn't bipartite, A cannot be an independent set and so we get our desired edge.

3. No clue. However, part 2 implies that this is true if G is not bipartite.

□

Problem 7. If you read the supplementary notes on Dyck paths, we proved that there are at most 4^n non-isomorphic trees on n vertices. This exercise will establish that there are at least α^n many non-isomorphic trees on n vertices for some $\alpha > 1$.

1. Prove that there are at least $n^{n-2}/n!$ many non-isomorphic trees on n vertices.

2. Use the inequality $1 - x \leq e^{-x}$ (which can be proved via elementary calculus if you care to do so) to prove that $n! \leq n^n / e^{(n-1)/2}$.

3. Conclude that there are at least $e^{(n-1)/2}/n^2$ many non-isomorphic trees on n vertices, which is approximately 1.6487^n for large n.

N.b. With a more careful upper-bound on $n!$ which can be found by approximating $\log n!$ by an integral (see Stirling's approximation), one can improve the lower-bound to approximately $e^n \approx 2.7182^n$. As mentioned in the supplementary notes, the actual answer is approximately 2.9557^n.

Solution.

1. Use Cayley's formula and the fact that isomorphisms are bijections.

2. Write $n! = \prod_{i=0}^{n-1}(n-i) = n^n \prod_{i=0}^{n-1}\left(1 - \frac{i}{n}\right)$ and apply the given inequality to each term in the product.

3. Profit!

□

Problem 8. Fix an integer $n \geq 2$ and let $d_1, \ldots, d_n$ be a sequence of positive integers with $\sum_{i=1}^n d_i = 2n-2$. Prove that the number of (labeled) trees T with vertex-set $[n]$ and $\deg_T i = d_i$ for all $i \in [n]$ is precisely

$$\binom{n-2}{d_1 - 1, \ldots, d_n - 1}.$$[1]

Solution. How many Prüfer codes are there wherein each i appears exactly $d_i - 1$ times? □

Problem 9. Let G be a connected graph and let $w\colon E(G) \to \mathbb{R}$ be a weight function. Consider the following "reverse-Kruskal algorithm":

Initialize $H = G$ and iterate the following process:

1. If H is acyclic, terminate and return H.

2. If H has a cycle, do the following. Let $\mathcal{C} \subseteq E(H)$ denote the set of all edges of H contained within a cycle of H. Take any edge $e \in \mathcal{C}$ of maximum weight, replace H by $H - e$ and repeat.

Prove that "reverse-Kruskal" returns a minimum spanning tree of G.

Solution. We know that the algorithm will eventually return a spanning tree of G since we only ever deleted edges contained within cycles and G is connected.

The proof that we get a minimum spanning tree is very similar to our proof of Kruskal. Indeed, take a minimum spanning tree T which shares the maximum number of edges with our tree. If T is not our tree, then, at some point in our algorithm, we must have deleted an edge of T; argue that this is impossible based on the definition of T. □

Problem 10. Let $T_1 \neq T_2$ be two trees on the same vertex set. For any edge $e \in E(T_1) \setminus E(T_2)$, we know that $T_2 + e$ contains a unique cycle; call this cycle C_e. Prove that

$$E(T_2) \setminus E(T_1) \subseteq \bigcup_{e \in E(T_1)\setminus E(T_2)} E(C_e).$$

[1] For any non-negative integers $k_1, \ldots, k_\ell$ with $\sum_{i=1}^\ell k_i = r$,

$$\binom{r}{k_1, \ldots, k_\ell} = \left|\left\{(A_1, \ldots, A_\ell) \in \left(2^{[r]}\right)^\ell : [r] = \bigsqcup_{i=1}^{\ell} A_i \text{ and } |A_i| = k_i\right\}\right| = \frac{r!}{k_1! \cdots k_\ell!}.$$

If you haven't seen multinomial coefficients before, convince yourself that $\binom{n}{k, n-k} = \binom{n}{k}$ as a warm-up.

Solution. Suppose that the common vertex-set of T_1 and T_2 is V. Fix any $s \in E(T_2) \setminus E(T_1)$; then $T_2 - s$ is disconnected. So we can find a partition $V = A \sqcup B$ such that both A and B are non-empty and $T_2 - s$ has no edge crossing between A and B. In particular, since T_2 is connected the *only* edge of T_2 which crosses between A and B is the edge s. Now, since T_1 is connected, we can find some $e \in E(T_1)$ which crosses between A and B; prove that $e \in E(T_1) \setminus E(T_2)$ and that $s \in E(C_e)$. □

Problem 11. Let G be a connected graph and let $w\colon E(G) \to \mathbb{R}$ be a weight function. Let $\mathcal{T}$ denote the set of all spanning trees of G and let $\mathcal{T}_{\min}$ denote the set of all minimum spanning trees of G.

1. Fix any $T_1 \in \mathcal{T}_{\min}$ and any $T_2 \in \mathcal{T}$ with $T_1 \neq T_2$. Prove that there is some $e \in E(T_2) \setminus E(T_1)$ and some $s \in E(T_1) \setminus E(T_2)$ such $T_3 = T_2 - e + s$ is a spanning tree of G and $w(T_3) \leq w(T_2)$.

2. Let $\mathcal{G}$ be the graph with vertex-set $\mathcal{T}$ where $T_1T_2 \in E(\mathcal{G})$ iff $|E(T_1) \triangle E(T_2)| = 2$.[2] Fix any $T \in \mathcal{T}$ and any $T' \in \mathcal{T}_{\min}$. Prove that there is a path $(T = T_0, \ldots, T_k = T')$ in $\mathcal{G}$ such that $w(T_i) \leq w(T_{i-1})$ for all $i \in [k]$.

3. Prove that $\mathcal{G}$ is a connected graph.

4. Let $\mathcal{G}_{\min}$ be the subgraph of $\mathcal{G}$ induced by $\mathcal{T}_{\min}$. Prove that $\mathcal{G}_{\min}$ is a connected graph.

Solution.

1. Since $T_1 \neq T_2$, we can find some $s \in E(T_1) \setminus E(T_2)$; let s be a minimum-weight edge in $E(T_1) \setminus E(T_2)$. Now $T_2 + s$ contains a unique cycle C which uses the edge s. Of course, C must have some edge e with $e \in E(T_2) \setminus E(T_1)$. Set $T_3 = T_2 - e + s$, so certainly T_3 is a spanning tree of G. If $w(e) \geq w(s)$, then $w(T_3) \leq w(T_2)$ and so we are done. Else, $w(e) < w(s)$; but then consider $T_1 + e$ and reach a contradiction to the fact that T_1 is a minimum spanning tree of G.

2. Show that $|T_1 \triangle T_2| = 2$ if and only if there is some $e \in E(T_1) \setminus E(T_2)$ and some $s \in E(T_2) \setminus E(T_1)$ such that $T_2 = T_1 - e + s$. Then use part 1.

3. Follows directly from part 2.

4. Follows directly from part 2.

□

Problem 12. Let G be a connected graph and let $w\colon E(G) \to \mathbb{R}$ be a weight function. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a strictly increasing function (that is, $x < y \iff f(x) < f(y)$) and consider a new weight function $w'\colon E(G) \to \mathbb{R}$ defined by $w' = f \circ w$. Prove that T is a minimum spanning tree with respect to w if and only if T is a minimum spanning tree with respect to w'.

Note: A naïve idea is to try to show that $x_1 + \cdots + x_k \leq y_1 + \cdots + y_k$ if and only if $f(x_1) + \cdots + f(x_k) \leq f(y_1) + \cdots + f(y_k)$. But this is false; indeed, $1 + 4 < 3 + 3$, yet $1^3 + 4^3 > 3^3 + 3^3$ (note that x^3 is a strictly increasing function). Instead use the key idea from Problem 11.

[2] Recall that $A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$; that is, $A \triangle B$ is the set of elements which are in either A or in B but not in both.

Solution. Of course, $w(T-e+s) = w(T) - w(e) + w(s)$ and $w'(T-e+s) = w'(T) - w'(e) + w'(s)$. Thus, $w(T - e + s) < w(T)$ if and only if $w(s) < w(e)$ and $w'(T - e + s) < w'(T)$ if and only if $w'(s) < w'(e)$. Now use Problem 11 and the fact that f is strictly increasing, so $w(s) < w(e) \iff w'(s) < w'(e)$. □

Problem 13. Let G be a connected graph. A vertex $c \in V(G)$ is called a *center* of G if $d(c, u) \leq \lceil \text{diam}(G)/2 \rceil$ for all $u \in V(G)$ (understand why this definition is sensible).

1. Find an infinite collection of (non-isomorphic) connected graphs which have no center.
2. Prove that if T is a tree then T has a center. Furthermore, prove that:
   (a) If $\text{diam}(T)$ is even, then T has a unique center.
   (b) If $\text{diam}(T)$ is odd, then T has exactly two centers and these two centers form an edge of T.
3. (Please do part 1 before this part) A graph G is called *vertex-transitive* if for any pair of vertices $u, v \in V(G)$, there is some automorphism $f \in \text{Aut}(G)$ with $f(u) = v$. Prove that if G is a connected, vertex-transitive graph which is not a clique, then G has no center.

   N.b. If you know a bit of group theory, then you can construct a diverse collection of vertex-transitive graphs known as Cayley graphs.

Solution.

1. There are many, but the easiest examples are the cycles C_n for each $n \geq 4$. To see this, note that $\text{diam}(C_n) = \lfloor n/2 \rfloor$ and that for all $u \in V(C_n)$, there is some $v \in V(G)$ for which $d(u, v) = \text{diam}(C_n)$.
2. Let T be a tree on at least two vertices; show that if $d(u, v) = \text{diam}(T)$, then both u and v are leaves of T. Then let L be the set of leaves of T. Supposing that T has at least 3 vertices, show that
   (a) $\text{diam}(T - L) = \text{diam}(T) - 2$.
   (b) u is a center of T if and only if u is a center of $T - L$.

   Using these facts, prove the claim by induction on the diameter of T.
3. First, prove that if $f \in \text{Aut}(G)$, then $d(u, v) = d(f(u), f(v))$ for all $u, v \in V(G)$. Conclude that if G is vertex-transitive, then for every $u \in V(G)$, there is some $v \in V(G)$ for which $d(u, v) = \text{diam}(G)$. Then observe that $\lceil d/2 \rceil = d$ if and only if $d \in \{0, 1\}$.

□

Problem 14. Let G be any disconnected, spanning subgraph of K_n and suppose that $G_1, \dots, G_k$ are the connected components of G. Set $V_i = V(G_i)$; note that $k \geq 2$ since G is disconnected and that we could have $|V_i| = 1$ for some (or all) i's.

Let $\mathcal{S}$ denote the set of all (labeled) subgraphs H of K_n such that

- H is a connected, spanning subgraph of K_n, and
- G is a subgraph of H, and

- If C is a cycle of H, then C is actually a cycle of G (that is, H contains no additional cycles).

Prove that

$$|\mathcal{S}| = n^{k-2} \prod_{i=1}^{k} |V_i|.$$

Hint #1: Letting $\mathcal{T}_k$ denote the set of all (labeled) trees on vertex-set $[k]$, consider the function $f\colon \mathcal{S} \to \mathcal{T}_k$ defined by, for $H \in \mathcal{S}$ and $i \neq j \in [k]$, $ij \in E(f(H))$ if and only if H has an edge with one vertex in G_i and the other in G_j. (You will need to show that f is well-defined, i.e. $f(H)$ is indeed a tree)

Hint #2: Problem 8 and the multinomial theorem will be helpful:

$$\left(\sum_{i=1}^{\ell} x_i\right)^r = \sum_{\substack{d_1,\ldots,d_\ell \in \mathbb{Z}_{\geq 0}:\\ d_1+\cdots+d_\ell = r}} \binom{r}{d_1,\ldots,d_\ell} \prod_{i=1}^{\ell} x_i^{d_i}.$$

Hint #2 (alternate): Alternatively, define a version of Prüfer codes that encapsulate this situation (hint #1 will still be helpful).

Solution. Prove first that f is well-defined; that is, $f(H)$ is indeed a tree for all $H \in \mathcal{S}$. Additionally, prove that if $ij \in E(f(H))$, then H has exactly one edge between G_i and G_j.

Now, fix any $T \in \mathcal{T}_k$ and prove that

$$|f^{-1}(T)| = \prod_{i=1}^{k} |V_i|^{\deg_T i}.$$

Therefore,

$$|\mathcal{S}| = \sum_{T \in \mathcal{T}_k} |f^{-1}(T)| = \sum_{T \in \mathcal{T}_k} \prod_{i=1}^{k} |V_i|^{\deg_T i}.$$

Grouping trees in $\mathcal{T}_k$ together based on the sequence $(\deg_T 1, \deg_T 2, \ldots, \deg_T k)$ and applying Problem 8 then yields

$$\begin{aligned}
|\mathcal{S}| &= \sum_{\substack{d_1,\ldots,d_k \in \mathbb{Z}_{\geq 1}:\\ d_1+\cdots+d_k = 2k-2}} \binom{k-2}{d_1-1,\ldots,d_k-1} \prod_{i=1}^{k} |V_i|^{d_i} \\
&= \left(\sum_{\substack{d_1,\ldots,d_k \in \mathbb{Z}_{\geq 0}:\\ d_1+\cdots+d_k = k-2}} \binom{k-2}{d_1,\ldots,d_k} \prod_{i=1}^{k} |V_i|^{d_i} \right) \prod_{i=1}^{k} |V_i| \\
&= \left(\sum_{i=1}^{k} |V_i| \right)^{k-2} \prod_{i=1}^{k} |V_i| = n^{k-2} \prod_{i=1}^{k} |V_i|.
\end{aligned}$$

For the alternate approach, we still need that f is well-defined and that if $ij \in E(f(H))$, then H has exactly one edge between G_i and G_j. To define the necessary version of Prüfer codes, consider deleting the smallest leaf ℓ from $f(H)$. Assuming that the unique neighbor of ℓ is j in $f(H)$, record the two endpoints of the edge of H which connected G_ℓ to G_j in two different lists. The first list will contain the vertex from G_j and the second list will contain the vertex from G_ℓ. Repeat until $f(H)$ has only a single edge left; then record the two vertices of G connected by this edge in the second list. The kicker is that the second list contains exactly one vertex from each V_i and we don't need the order in order to reconstruct H (why?). Thus, this process will exhibit a bijection between $\mathcal{S}$ and $[n]^{k-2} \times V_1 \times \cdots \times V_k$ (replicate the proof that Prüfer codes work in order to show this). $\square$