These solutions are from https://mathematicaster.org/teaching/graphs2022/sol-ds4.pdf

I encourage you to first read through all of these problems and then focus first on those with which you're less comfortable.

Problem 1. You and your friends want to tour the American southwest by car (see the picture below). For some strange reason (probably since you're all nerds and are taking this class :)), you decide you wish to cross the border between every pair of neighboring states[1] exactly once (and not visit any states not in the southwest). Can you accomplish this peculiar goal? If so, does it matter where you start/end your road trip? Justify your answer.

Solution. It is possible, and any such trip must start in either Nevada or Utah and end in the other. To prove this, build an auxiliary graph which encapsulates the problem and argue based on Eulerian circuits/trails. □

Problem 2. Let G be a bipartite graph with parts A, B where $|A| \geq |B|$. Of course, $\alpha'(G) \leq |B|$. Furthermore, since A is an independent set, certainly we have $\alpha(G) \geq |A|$.

Prove that $\alpha(G) = |A|$ if and only if $\alpha'(G) = |B|$.

Solution. Kőnig + HW9.4. □

Problem 3. Let $G = (V, E)$ be a graph. We say that two non-empty sets $A, B \subseteq V$ (which could intersect) form a *disjunction* of G if G has no edge with one end-point in A and the other in B. Note that disjunctions always exist since we could take $A = B = \{v\}$ for some vertex v.

Prove that for any graph G (yes, even cliques),

$$\kappa(G) = |V| - \max_{A,B \text{ a disjunction of } G} |A \cup B|.$$

Solution. First handle cliques: prove A, B is a disjunction if and only if $A = B = \{v\}$ for some vertex v.

For a non-clique, prove two inequalities. For one, show that if U is a vertex-cut, then we can partition $V \setminus U = A \sqcup B$ so that A, B is a disjunction. For the other inequality, show that if A, B is a disjunction with $|A \cup B| \geq 2$, then $V \setminus (A \cup B)$ is a vertex-cut (hint: show that we may suppose that A and B are disjoint in this case). □

Problem 4. We have a standard deck of 52 cards, so there are 4 suits and 13 ranks. We deal the cards (arbitrarily) into 13 piles, each containing four cards. Prove that we can select exactly one card from each pile so that we wind up with one card of each rank.

[1] We do not consider Utah and New Mexico nor Colorado and Arizona to be neighboring states since they touch only at a point.

Solution. We can model this by a bipartite *multi*graph G where the left-set is the ranks and the right-set is the piles. Note that G really is a multigraph since a pile can contain multiple cards of the same rank. As should be apparent, we're looking for a perfect matching in G.

The good thing is that the bipartite handshaking lemma is still true for multigraphs! So actually Theorem 2 from 03-31 holds for multigraphs as well since we only relied on the handshaking lemma and never used $\deg x = |N(x)|$. □

Problem 5. Show that any connected graph contains a walk which traverses each edge *exactly* twice.

(Extra fun: Prove this both with and without invoking multigraphs)

Solution. If you like multigraphs, consider replacing each edge by two edges with the same end-points. Then apply what you know about Eulerian circuits and interpret such a circuit as a walk on the original graph.

If you don't like multigraphs, break into two cases:

- If G is even-regular, simply follow an Eulerian circuit around twice.
- Otherwise, consider taking two copies of G and placing an edge between the copies of a vertex iff it has odd degree. Then apply what you know about Eulerian circuits and interpret such a circuit as a walk on the original graph.

□

Problem 6. Fill in the "BLAH" in the following statement (and prove it): If G is a connected graph, then G contains a walk which traverses each edge *exactly* thrice if and only if BLAH.

(Hint: You probably want to use multigraphs here even though G is simple)

Solution. BLAH="G has an Eulerian circuit or trail".

The leftward direction is clear since we can simply follow an Eulerian circuit or trail there, back and there again.

The rightward direction is more interesting.

Consider replacing each edge by three edges with the same end-points. Then show that such a walk can be transformed into an Eulerian trail or circuit on this new graph. Then interpret what this means about the original degrees of the graph.

Now, I'll be honest, I haven't carefully checked that the following works, but I believe it should (though I could easily be wrong):

If you don't like multigraphs, suppose that G has neither an Eulerian circuit nor trail. Build a new graph by taking three copies of G and placing edges between the copies of a vertex iff it has odd degree. Show that the walk on G can be transformed into an Eulerian circuit/trail on this new graph. Then interpret what this means about the original degrees of the graph. □

Problem 7. There are a bunch of islands connected by a bunch of bridges. You are given a torch and charged with burning all of these bridges. Of course, you can only burn bridges which touch the island you're currently standing on and you can't cross a bridge that you've previously burned. Furthermore, if you ever cross a bridge, you must burn that bridge behind you.

Using objects/terms that we've been discussing lately, determine exactly when you can burn all of the bridges assuming that you get to decide on which island you begin your rampage.

(Note: I don't know if there are reasonable necessary/sufficient conditions on, say, the degrees of this island-bridge network which work, though I doubt it.)

Solution. We can model the island-bridge situation by a graph G.

Answer: this is possible if and only if G contains a trail (either open or closed) T such that $V(T)$ is a vertex-cover of G.

I don't know if there are reasonable necessary/sufficient conditions on, say, the degrees of G which imply such a trail exist. □

Problem 8. A *Latin square of order* n is a matrix $A \in [n]^{n\times n}$ such that every $i \in [n]$ appears exactly once in each row and each column of A. Note that any Sudoku is a Latin square of order 9, although there are additional rules in Sudoku.

Suppose that $B \in [n]^{m\times n}$ for some $m < n$ and each $i \in [n]$ appears exactly once in each row of B and at most once in each column of B. Prove that B can be extended to a Latin square of order n (i.e. one can fill in the remaining $n - m$ rows of B to create a valid Latin square).

(Hint: Just show that one can fill in the $(m+1)$st row and then conclude by induction on $n-m$)

(Hint: Relate the problem of filling in the $(m + 1)$st row to one of finding a system of distinct representatives (or directly to finding a matching in a bipartite graph))

Solution. For each $i \in [n]$, let S_i denote the set of elements of $[n]$ which do not appear in the ith column of B. Then filling in the $(m+1)$st is equivalent to finding a system of distinct representatives of $S_1, \dots, S_n$. To show that such an SDR exists, show that $|S_i| = n - m$ for each $i \in [n]$ and also that each $i \in [n]$ appears in exactly $n - m$ of the S_j's. □

Problem 9. Suppose that G is a graph with no isolated vertices.

1. Show that if G is connected and has no copy of P_4, then either $G \cong K_3$ or $G \cong K_{1,n}$ for some $n \geq 1$.
2. Suppose that $S \subseteq E(G)$ is a minimum edge-cover of G, so $|S| = \beta'(G)$. Prove that (V, S) is a forest with no isolated vertices wherein each connected component is a star (i.e. a copy of $K_{1,n}$ for some $n \geq 1$).

Solution.

1. Show that such a G must have the property that for every $e, s \in E(G)$, we have $e \cap s \neq \varnothing$.

   Now, work through $|V(G)| \leq 3$ by hand. Then, for $|V(G)| \geq 4$, show that every pair of edges intersect if and only if all edges contain some fixed, common vertex. Conclude that G must be a star.

2. We already showed that (V, S) is a forest with no isolated vertices. Show also that (V, S) cannot contain a copy of P_4; then apply part 1 to the connected components of (V, S).

□

Problem 10. Let $G = (V, E)$ be a graph. For a function $f : E \to \{0, 1\}$, $i \in \{0, 1\}$ and $v \in V$, we define

$$\deg_i^f v = |\{e \in E : v \in e \text{ and } f(e) = i\}|.$$

Observe that $\deg_0^f v + \deg_1^f v = \deg v$ for any $v \in V$.

1. Suppose that G is an even-regular graph. Show that if $|E|$ is odd and $f : E \to \{0, 1\}$ is any function, then there must be some $v \in V$ such that $|\deg_0^f v - \deg_1^f v| \geq 2$.

2. Suppose that G is a connected, even-regular graph and fix any $v^* \in V$. Prove that there is a function $f\colon E \to \{0,1\}$ satisfying

   (a) $\deg_0^f v = \deg_1^f v$ for all $v \in V \setminus \{v^*\}$, and

   (b) If $|E|$ is even, then also $\deg_0^f v^* = \deg_1^f v^*$, and

   (c) If $|E|$ is odd, then $|\deg_0^f v^* - \deg_1^f v^*| = 2$.

3. Suppose that G is a connected graph with at least one odd-degree vertex. Prove that there is a function $f\colon E \to \{0,1\}$ such that $|\deg_0^f v - \deg_1^f v| \leq 1$ for all $v \in V$.

N.b. It turns out that the sorta converse holds: If G is connected and $f\colon E \to \{0,1\}$ is a function such that $\deg_0^f v = \deg_1^f v$ for all $v \in V$, then G has an Eulerian circuit $(v_0, e_1, \ldots, v_{m-1}, e_m, v_m = v_0)$ such that $f(e_i) = i \mod 2$ for all $i \in [m]$. Proving this fact requires proving analogues of all of the lemmas needed to establish the existence of Eulerian circuits. This is a good exercise if you care to put in the work.

Solution.

1. First show that the conclusion fails to hold if and only if $\deg_0^f v = \deg_1^f v$ for all $v \in V$ since G is even-regular. Then, use the handshaking lemma to show that $|f^{-1}(0)| = |f^{-1}(1)|$. Use this to conclude that G has an even number of edges.

2. Since G is connected and even-regular, we know G has an Eulerian circuit. Label this circuit as $(v_0, e_1, v_1, e_2, \ldots, v_{m-1}, e_m, v_m = v_0)$ (so $|E| = m$). Since this is a circuit, we may suppose that $v_0 = v^*$. Then define $f(e_i) = i \mod 2$. Prove that this f satisfies the claim.

3. Add a new vertex v^* connected to all odd-degree vertices of G. Then apply part 2 to this new graph with v^* being the special vertex. Then show that by deleting v^*, we get the desired function.

   This is one case where "adding a matching" between the odd-degree vertices won't work.

□