These solutions are from https://mathematicaster.org/teaching/graphs2022/sol-ds5.pdf

I encourage you to first read through all of these problems and then focus first on those with which you're less comfortable.

Problem 1. For each odd integer $n \geq 1$, construct a graph G on n vertices for which both $\chi(G)$ and $\chi(\overline{G})$ are at least $(n+1)/2$.

This shows that the Nordhaus–Gaddum inequalities (Problem 2) are tight.

Solution. Handle the case of $n = 1$ first (only one thing you can do).

For $n \geq 3$, start with disjoint A, B where $|A| = (n+1)/2$ and $|B| = (n-1)/2$. Form G by placing a clique in A and not having any other edges. Then show that $\overline{G}$ also contains a clique on at least $(n+1)/2$ many vertices. □

Problem 2 (Nordhaus–Gaddum inequalities)**.** Let G be a graph on n vertices. Prove that

$$\chi(G) \cdot \chi(\overline{G}) \leq \frac{(n+1)^2}{4}, \quad \text{and}$$
$$\chi(G) + \chi(\overline{G}) \leq n + 1.$$

(Technically, the second inequality implies the first, but I think it's worth stating both of them)

The key idea behind both inequalities is to relate the degeneracy of G to that of $\overline{G}$. In particular, prove that $d(\overline{G}) \leq n - d(G) - 1$ and then use the fact that $\chi(H) \leq d(H) + 1$ to derive the stated inequalities.

Road map for showing that $d(\overline{G}) \leq n - d(G) - 1$:

1. Let H be a subgraph of G with $\delta(H) = d(G)$ and let H' be a subgraph of $\overline{G}$ with $\delta(H') = d(\overline{G})$.
2. Suppose for the sake of contradiction that $d(\overline{G}) \geq n - d(G)$ and argue that $V(H) \cap V(H') = \varnothing$.
3. Reach a contradiction by comparing $|V(H)|$ and $|V(H')|$.

Solution. For the road-map, the key observation is that if $v \in V(H) \cap V(H')$, then $\deg_G v \geq \delta(H)$ and $\deg_{\overline{G}} v \geq \delta(H')$, which is rendered impossible if $\delta(H') \geq n - d(G) = n - \delta(H)$.

To reach the contradiction in the last step of the road-map, note that $|V(H)| \geq \delta(H) + 1$ and $|V(H')| \geq \delta(H') + 1$.

Once you've shown that $d(\overline{G}) \leq n - d(G) - 1$, then you have $\chi(G) \leq d(G) + 1$ and $\chi(\overline{G}) \leq n - d(G)$. Thus, $\chi(G) + \chi(\overline{G}) \leq n + 1$ is immediate. To show that $\chi(G) \cdot \chi(\overline{G}) \leq (n+1)^2/4$, you need to note that $(x+1)(n-x) \leq (n+1)^2/4$ for any x. The easiest way to show this is probably the AM–GM inequality: $\sqrt{xy} \leq (x+y)/2$ (which can be proved by expanding $(x-y)^2 \geq 0$), but you could also do some quick optimization via calc I yumminess if so desired. □

Problem 3. Let D be a digraph with no loops. We define proper vertex-colorings of a digraph to be the same as proper vertex-colorings of its underlying simple graph (so we just forget about directions). In particular, $\chi(D)$ is the same as $\chi(G)$ where G is the underlying simple graph of D.

Let $p(D)$ denote the number of *vertices* in a longest directed path in D (recall that (x) is always a dipath which has 1 vertex).

1. Suppose that D is acyclic (has no directed cycles, though the underlying simple graph could have cycles). Prove that $\chi(D) \le p(D)$.

   Hint: Let $f(v)$ denote the number of vertices in a longest dipath which ends at v. Show that f is a proper $p(D)$-coloring of D.

2. Show that part 1 still holds even if D contains dicycles.

   Hint: Take a maximally acyclic subgraph of D and apply the hinted f to this subgraph. Then show that every edge which was deleted when reducing to this subgraph is also properly colored under f.

3. A *tournament* of order n is simply an orientation of K_n. Show that every tournament contains a Hamiltonian dipath (a dipath which contains all vertices).

4. Let T be a tournament of order n and consider coloring the edges of T red and blue. Prove that T contains a monochromatic (all edges the same color) dipath on at least $\sqrt{n}$ many vertices.

Solution.

1. As suggested, define $f(v)$ to be the number of vertices in a longest dipath ending at v. Observe that $f(v) \in [p(D)]$. Given any edge $(x, y) \in E(D)$, show that $f(y) \ge f(x) + 1$ by slapping y onto the end of a longest dipath ending at x. Here, it is very important that D is acyclic so that we know that y didn't appear earlier in such a dipath (why does the acyclicness imply this?)

2. Letting H be a maximally acyclic digraph of D, we know that for any $e \in E(D) \setminus E(H)$, the digraph $H + e$ must contain a dicycle. So if $e = (u, v)$, then such a dicycle must look like $(v = v_1, v_2, \dots, v_k = u)$. If f is the coloring of H in the hint from part 1, then we would have $f(v_1) < f(v_2) < \cdots < f(v_k)$, so $f(u) \ne f(v)$.

   Thus, f is also a proper coloring of D and so $\chi(D) \le p(H) \le p(D)$.

3. $\chi(K_n) = n$, so apply part 2.

4. HW11.2 + part 2.

□

Problem 4. Let G be a connected plane graph and suppose that every face of G has length either 5 or 6. If G is additionally 3-regular, show that G must have exactly 12 faces of length 5.

So it is no accident that soccer balls have exactly 12 pentagons on their surface!

Solution. Let P be the set of length-5 faces and let H be the set of length-6 faces, so $|F| = |P| + |H|$ by assumption. The headshaking lemma tells us that

$$2|E| = \sum_{f \in F} \operatorname{len}(f) = 5|P| + 6|H|.$$

Then, since G is 3-regular, the handshaking lemma tells us that

$$3|V| = \sum_{v \in V} \deg v = 2|E| = 5|P| + 6|H|.$$

Now, G is connected, so we may apply Euler's formula to find that

$$2 = |V| + |F| - |E| = \frac{1}{3}\big(5|P| + 6|H|\big) + \big(|P| + |H|\big) - \frac{1}{2}\big(5|P| + 6|H|\big) = \frac{1}{6}|P|.$$

□

Problem 5. Let G be a connected plane graph wherein every face is bounded by a cycle. Prove that if G has no cycles of length 5 or shorter, then $\chi(G) \leq 3$.

Is there any bound on the cycle lengths (e.g. forbidding all cycles of length less than $10^{10^{10^{10}}}$) that would imply that $\chi(G) \leq 2$?

Solution. Use HW12.2 to show that $d(G) \leq 2$ and then apply our greedy coloring bound.

There is no such bound that would imply that $\chi(G) \leq 2$. Even if we forbid all cycles of length $\leq g$, then, no matter how large g is, there will always be an odd integer k with $k > g$. Then C_k is an odd cycle, which has chromatic number 3 and is also planar. □

Problem 6. Is there a graph G on exactly 6 vertices which is non-planar, yet does not contain a copy of K_5 nor $K_{3,3}$?

Solution. Yup, just subdivide one edge of K_5 once. This is, however, the only such graph. □

Problem 7. Let G be a graph. G contains vertices $v_1, \ldots, v_5$ where $\deg v_1 = 100$, $\deg v_2 = 30$, $\deg v_3 = 30$, $\deg v_4 = 4$, $\deg v_5 = 3$ and all other vertices of G have degree either 1 or 2. Knowing nothing else about G, can you determine whether or not G is planar?

Solution. What can you say about the degrees of G if it contained a subdivision of K_5 or of $K_{3,3}$? Then reference Kuratowski. □

Problem 8. The *crossing number* of G, denoted by $\mathrm{cr}(G)$ is the minimum number of pairs of edges of G that must cross when attempting to draw G in the plane. In particular $\mathrm{cr}(G) = 0$ iff G is planar. Similarly $\mathrm{cr}(G) = 1$ iff G is non-planar and there is a drawing of G in which exactly two of the edges cross (since cr counts *pairs* of crossing edges).

1. Show that $\mathrm{cr}(K_5) = \mathrm{cr}(K_{3,3}) = 1$.
2. Suppose that G is a graph with $n \geq 3$ vertices and m edges. Prove that $\mathrm{cr}(G) \geq m - 3n + 6$.

   (To make life easier, feel free to assume that Theorem 9 from 04-14 holds even if G is disconnected (it does still hold provided $n \geq 3$; we just didn't prove it))

Solution.

1. It's not too difficult to come up with such drawings. Try your best and you'll probably succeed.
2. Take a drawing of G with the minimum number of crossings, so it has exactly $\mathrm{cr}(G)$ many pairs of crossing edges. Now, let G' be the subgraph of G formed by deleting one edge per crossing of G. By construction, G' is planar (since we just created a planar drawing of it by removing these misbehaving edges), has n vertices and $m - \mathrm{cr}(G)$ many edges. Since $n \geq 3$, we know that $m - \mathrm{cr}(G) = |E(G)| \leq 3n - 6 \implies \mathrm{cr}(G) \geq m - 3n + 6$.

□