These solutions are from https://mathematicaster.org/teaching/graphs2022/sol-ds6.pdf

I encourage you to first read through all of these problems and then focus first on those with which you're less comfortable.

Problem 1. Let n be any positive integer. Set $N = 2n$ if n is odd and set $N = 2n - 1$ if n is even. Show that every red,blue-coloring of $E(K_N)$ contains a monochromatic copy of $K_{1,n}$.

Solution. Letting $\deg_r v$ and $\deg_b v$ be the red- and blue-degrees of the vertex v in a coloring, you just need to show that there's some vertex for which either $\deg_r v \geq n$ or $\deg_b v \geq n$. If n is odd (and so $N = 2n$), then this is immediate from the fact that $\deg_r v + \deg_b v = 2n - 1$. If n is even (and so $N = 2n - 1$), then $\deg_r v + \deg_b v = 2n - 2$, so the only way this can fail is if $\deg_r v = \deg_b v = n - 1$ for all vertices v. But both $n - 1$ and N are odd, so this is impossible (odd number of odd degrees!). □

Problem 2. Show that if G is 3-connected, then any pair of vertices $u \neq v \in V(G)$ are contained together within an even cycle.

Solution. Menger tells us that there are three internally disjoint u-v paths. Combining any two of these paths yields a cycle containing u and v. Now, since we have three of these paths, some pair have lengths of the same parity, so we get an even cycle by combining those. □

Problem 3. Let G be a bipartite graph with parts A, B which has at least one edge and set $D = \{v \in V(G) : \deg v = \Delta(G)\}$. Prove that G contains a matching which saturates $A \cap D$.

(Note: if $A \cap D = \varnothing$, then the "empty matching" suffices)

Solution. If $A \cap D = \varnothing$ then we are done. Otherwise, consider the subgraph G' induced on $A \cap D$ and B. Then, in G', every vertex in $A \cap D$ has degree exactly $\Delta(G)$ (since we kept all of their neighbors) and every vertex in B has degree at most $\Delta(G)$. Since $\Delta(G) \geq 1$ (since G has an edge), the claim then follows by applying Corollary 4 from 03-31 to G'. □

Problem 4. For an integer $N \geq 2$, let K_N^- denote the graph formed by deleting exactly one edge from K_N (it really doesn't matter which edge since they're all identical).

Fix any integer $n \geq 2$ and set $N = R(n, n)$. By definition, every red,blue-coloring of $E(K_N)$ contains a monochromatic copy of K_n. However, prove that there exists some red,blue-coloring of $E(K_N^-)$ which *does not* contain a monochromatic copy of K_n. In other words, show that every single edge of K_N is important when it comes to forcing a monochromatic copy of K_n.[1]

Solution. By definition, we can find a red,blue-coloring of $E(K_{N-1})$ which does not contain a monochromatic K_n. Lift this to a coloring of $E(K_N^-)$ by "duplicating" a vertex. This works since a clique cannot use both the original and duplicated vertex since there's no edge between them! □

Problem 5. Recall that the 3-color Ramsey number $R(m, n, p)$ is the smallest integer N such that every 3-coloring of $E(K_N)$ (say with colors red,blue,green) contains either a red K_m, a blue K_n or a green K_p.

[1] N.b. If you study more Ramsey theory in the future, you may come across the so-called "size-Ramsey numbers" $\widehat{R}(H)$, which is the fewest number of edges in a graph G such that any 2-coloring of $E(G)$ has a monochromatic copy of the graph H. This problem essentially proves that $\widehat{R}(K_n) = \binom{R(n,n)}{2}$.

1. Show that if $m, n, p \geq 2$, then
$$R(m, n, p) \leq R(m-1, n, p) + R(m, n-1, p) + R(m, n, p-1) - 1.$$

2. Show that if $m, n, p \geq 1$, then
$$R(m, n, p) \leq R\big(m, R(n, p)\big).$$

Can you see how either inequality proves that $R(m, n, p)$ actually exists for all m, n, p? Can you see how to generalize both inequalities to the "t-color Ramsey number"?

Solution.

1. Run the exact same argument from class that showed that $R(m, n) \leq R(m-1, n) + R(m, n-1)$ except with three colors.

2. Make yourself color-blind for a moment and pretend you can't distinguish blue and green.

□

Problem 6. Let $C(n)$ denote the set of all red,blue-colorings of $E(K_n)$. Show that

$$\operatorname*{average}_{f \in C(n)} \#\{\text{mono}\chi \text{ triangles in } f\} = \frac{1}{4}\binom{n}{3}.$$

Conclude that there is some red,blue-coloring of $E(K_n)$ in which *strictly* fewer than 1/4 of all triangles are monochromatic.

This shows that the theorem we proved in class is (approximately) tight.

Solution. Run almost exactly the same proof we did in class by exploiting indicator functions and switching the order of summation. You'll then be left with determining the number of ways to color $E(K_n)$ so that some fixed triangle is monochromatic. Once you figure that out, the equality just pops out!

Now, not everyone can be above average, so certainly there's a coloring with *at most* $\frac{1}{4}\binom{n}{3}$ many monochromatic triangles. *However*, if there's even one person that's strictly above average, then there also must be someone who's strictly below average! And giving every edge the same color yields a coloring which has strictly more monochromatic triangles than the average! □

Problem 7.

1. Show that every red,blue-coloring of $E(K_n)$ must contain a monochromatic tree on n vertices

2. Let T be any tree on t vertices, let n be a positive integer and set $N = n + t - 1$. Prove that any red,blue-coloring of $E(K_N)$ contains either a red copy of T or a blue copy of $K_{1,n}$. (This generalizes part of Problem 1.)

3. Consider any n which is a multiple of 4. Construct a 3-coloring of $E(K_n)$ which *does not* contain a monochromatic tree on strictly more than $\frac{n}{2}$ many vertices.

Solution.

1. Either G or $\overline{G}$ is connected.

2. There is a blue $K_{1,n}$ if and only if $\deg_b v \geq n$ for some vertex v. If this is not the case, then $\deg_b v \leq n-1 \implies \deg_r v = N-1-\deg_b v = n+t-2-\deg_b \geq t-1$. So the red-graph has min degree at least $t-1$ and so it must contain a copy of T.

3. Start with a proper 3-edge-coloring of K_4 and "blow-up" each vertex into a cluster of $n/4$ many vertices (color any edges within these chunks arbitrarily). Then any connected component of any color-class sees at most two of these chunks, so any monochromatic tree sees at most two of these chunks as well.

□

Problem 8. Show that any t-coloring of $E(K_n)$ contains a monochromatic tree on at least $n/(t-1)$ many vertices.

Roadmap:

1. Start by showing that if G is a bipartite graph with parts A, B, then G contains a tree on at least $\left(\frac{1}{|A|}+\frac{1}{|B|}\right)\cdot|E|$ many vertices. In particular, show it contains a "double star" of this size, where a double star is two stars whose centers are connected by an edge.

   (a) Use Cauchy–Schwarz[2] and the bipartite handshaking lemma to show that

   $$\sum_{\substack{a\in A, b\in B:\\ ab\in E}} (\deg a + \deg b) \geq \left(\frac{1}{|A|}+\frac{1}{|B|}\right)\cdot|E|^2.$$

   (b) Why does this give you the desired double star?

2. Now consider a t-coloring of $E(K_n)$. If the color-t-graph is connected, then we win. Otherwise, the color-t-graph has a break; apply part 1 to this break in some way.

Solution. For the first part of the roadmap, we employ Cauchy–Schwarz and the fact that G is bipartite to bound

$$\begin{aligned}\sum_{\substack{a\in A, b\in B:\\ ab\in E}} (\deg a + \deg b) &= \sum_{\substack{a\in A, b\in B:\\ ab\in E}} \deg a + \sum_{\substack{a\in A, b\in B:\\ ab\in E}} \deg b = \sum_{a\in A}\deg^2 a + \sum_{b\in B}\deg^2 b\\ &\geq \frac{1}{|A|}\left(\sum_{a\in A}\deg a\right)^2 + \frac{1}{|B|}\left(\sum_{b\in B}\deg b\right)^2 = \left(\frac{1}{|A|}+\frac{1}{|B|}\right)\cdot|E|^2.\end{aligned}$$

Since not everyone can be below average, this means that there must be some edge $xy \in E$ where $x \in A, y \in B$ for which

$$\deg x + \deg y \geq \frac{1}{|E|}\sum_{\substack{a\in A, b\in B:\\ ab\in E}} (\deg a + \deg b) \geq \left(\frac{1}{|A|}+\frac{1}{|B|}\right)\cdot|E|.$$

[2] in particular, the special case which stated: $\sum_{i=1}^n a_i^2 \geq \frac{1}{n}\left(\sum_{i=1}^n a_i\right)^2$

Taking all edges incident to x and to y then yields a double star (because G is bipartite) on $\deg x + \deg y$ many vertices.

For the second part, let $G_1, \ldots, G_t$ be such that G_i is the graph formed by the i-colored edges. If G_t is connected, then the coloring contains a spanning tree in color t, which is a monochromatic tree on $n \geq n/(t-1)$ many vertices as needed. Otherwise, G_t is disconnected, so we can find a partition $[n] = A \sqcup B$ with $A, B \neq \varnothing$ and G_t has no edges crossing between A and B. In particular, every edge between A and B is colored with only the colors in $[t-1]$. Thus, there is some color $i \in [t-1]$ which has at least $|A| \cdot |B|/(t-1)$ many edges crossing between A and B. Applying the first part of the roadmap to the bipartite graph formed by the color-i edges between A and B then yields a color-i tree on at least

$$\left(\frac{1}{|A|} + \frac{1}{|B|}\right) \cdot \frac{|A| \cdot |B|}{t-1} = \frac{|A| + |B|}{t-1} = \frac{n}{t-1}$$

many vertices as needed. □

Problem 9. In class, we showed that any sequence of n *distinct* real numbers contains a monotone subsequence of length $> \frac{1}{2}\log_2 n$. Use DS5.3.4 to improve this bound to $\geq \sqrt{n}$ (which is actually the correct answer).

Solution. Treat $V(K_n) = [n]$. Form a tournament T by orienting the edges of K_n from smaller to larger; that is $(i,j) \in E(T) \iff i < j$. Then color the edges of T based on the relative ordering of the elements of the sequence as we did in class. Finally, DS5.3.4 tells us that this coloring of $E(T)$ contains a monochromatic dipath of length $\geq \sqrt{n}$; use this to conclude that there's a monotone subsequence of this length. □

Problem 10. You just got a new TV, but the remote didn't come with any batteries and requires two batteries to operate... Luckily, you have a box of $2n \geq 4$ old batteries to fuel your new remote! You remember that exactly n of these batteries are completely dead and exactly n of these batteries have at least some charge. Unfortunately, they are scattered about and you can't tell which is which without testing them in your new remote. So, all you can do is insert two of these batteries into your remote and see if it works. The remote will not work at all if even one of the inserted batteries is dead. Determine (in terms of n) the fewest number of trials needed to make your remote work.

(To help you along, the correct answer is 6 if $n = 2$ and is $n + 3$ if $n \geq 3$.)

Solution. This is just a special case of Turán's theorem in disguise! Build a graph G where the vertex-set is the set of the $2n$ batteries where we place an edge if we tested two batteries together. Then we will be sure to succeed if and only if $\alpha(G) < n$ (or else it is possible that we did not test any pair of working batteries and our remote never worked). So our question is: What is the fewest number of edges in a $2n$-vertex graph which has no independent set of size n? Flipping to the complement: What is the maximum number of edges in a $2n$-vertex graph which has no clique of size n? The latter question is answered by Turán's theorem, so there you go!

If you actually want the optimal strategy, you should test batteries based on the *non*-edges of $T_{n-1}(2n)$ (Turán's theorem says that this is the *only* optimal strategy (up to isomorphism)). If $n = 2$, then $\overline{T_{n-1}(2n)} \cong K_4$ and if $n \geq 3$, then $\overline{T_{n-1}(2n)}$ consists of two triangles and a matching. □

Problem 11. Let T be any tree on $t \geq 2$ vertices.

1. Prove that $\mathrm{ex}(n, T) \geq \frac{1}{2}(t-2)n$ whenever $(t-1) \mid n$.

2. Prove that $\text{ex}(n, T) \le (t-2)n$.

   (Hint: Show that any graph G contains a subgraph H with $\delta(H) \ge |E(G)|/|V(G)|$. To show this, consider taking H to be a subgraph of G which maximizes the quantity $|E(H)|/|V(H)|$.)

N.b. A conjecture of Erdős and Sós from the 60's posits that $\text{ex}(n, T) \approx \frac{1}{2}(t-2)n$ where the "$\approx$" is simply a rounding error if $(t-1) \nmid n$.

Solution.

1. Consider the disjoint union of $n/(t-1)$ many copies of K_{t-1}.

2. First prove the hint. The main observation is that if H has at least two vertices and $\delta(H) < |E(H)|/|V(H)|$, then if $v \in V(H)$ has $\deg_H v = \delta(H)$, then

$$\frac{|E(H-v)|}{|V(H-v)|} = \frac{|E(H)| - \delta(H)}{|V(H)| - 1} > \frac{|E(H)| - |E(H)|/|V(H)|}{|V(H)| - 1} = \frac{|E(H)|}{|V(H)|}.$$

   With the hint out of the way, suppose that G is an n-vertex, T-free graph. If $|E(G)| > (t-2)n$, then G contains a subgraph H with $\delta(H) \ge |E(G)|/|V(G)| > t-2 \implies \delta(H) \ge t-1$. But then H contains a copy of T since T has t vertices; a contradiction.

$\square$