These solutions are from https://mathematicaster.org/teaching/graphs2022/sol-hw1.pdf

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (2 pts)**.** Let S be a finite set of integers and let G be the graph on vertex set S where for any $x, y \in S$, $xy \in E(G)$ if and only if $x + y$ is odd.

1. Prove that G is a bipartite graph for any such S.
2. If $S = [100]$, what is $|E(G)|$? Justify your answer.

Solution.

1. Observe that $xy \in E(G)$ if and only if x and y have different parities. Thus, if $A = \{x \in S : x \text{ is even}\}$ and $B = \{x \in S : x \text{ is odd}\}$, then $S = V(G) = A \sqcup B$ and every edge of G connects a vertex of A to a vertex of B. In particular, $A \sqcup B$ is a bipartition of G and so G is bipartite.
2. As mentioned in the previous part, $xy \in E(G)$ if and only if x and y have different paritites. Therefore, using the sets in the previous part, every vertex of A is adjacent to every vertex of B and so $|E(G)| = |A| \cdot |B|$. In the case when $S = [100]$, we have $|A| = |B| = 50$ and so $|E(G)| = 2500$.

□

Problem 2 (2 pts)**.** Suppose that $(u = v_0, v_1, \ldots, v_k = v)$ is a u-v geodesic. Prove that $d(u, v_i) = i$ for all $i \in \{0, \ldots, k\}$.

Solution. Firstly, since $(v_0, \ldots, v_k)$ is a u-v geodesic, we know that $d(u, v) = k$. Fix any $i \in \{0, \ldots, k\}$. Since $(u = v_0, \ldots, v_i)$ is a u-v_i path of length i, we know that $d(u, v_i) \leq i$. Thus, suppose for the sake of contradiction that $d(u, v_i) = \ell < i$; hence we may find a u-v_i geodesic $(u = w_0, \ldots, w_\ell = v_i)$. As such, $(u = w_0, \ldots, w_\ell = v_i, v_{i+1}, \ldots, v_k = v)$ is a u-v *walk* of length $\ell + (k - i)$. Theorem 1.6 then implies that there is a u-v path of length at most $\ell + (k - i) < k$ contradicting the fact that $d(u, v) = k$. □

Problem 3 (2 pts)**.** Let G be a graph. For two non-empty subsets $A, B \subseteq V(G)$, an A-B path is a path in G which connects some vertex of A to some vertex of B. Prove that if P is a minim**al** A-B path, then P contains exactly one vertex from A and contains exactly one vertex from B.

Solution. Let $P = (v_0, \ldots, v_k)$ be a minimal A-B path; so $v_0 \in A$ and $v_k \in B$. If $k = 0$ (which can happen only if A and B intersect), then we are trivially done, so we may suppose that $k \geq 1$. Suppose for the sake of contradiction that there was some $i \in \{1, \ldots, k\}$ such that $v_i \in A$ as well (note that i could equal k if A and B intersect). But then $(v_i, \ldots, v_k)$ is also an A-B path which is a proper sub-path of P; a contradiction to the minimality of P. Therefore P contains exactly one vertex (namely v_0) of A. A symmetric argument establishes that P contains exactly one vertex (namely v_k) of B. □

Problem 4 (2 pts)**.** Let G be a connected graph. Prove that any two maxim**um** paths in G must share some vertex.

Solution. Suppose for the sake of contradiction that $(v_0, \ldots, v_k)$ and $(u_0, \ldots, u_k)$ are two maximum paths in G which are vertex-disjoint (in particular, the length of any maximum path in G is k). Set $A = \{v_0, \ldots, v_k\}$ and $B = \{u_0, \ldots, u_k\}$; since G is connected, there is an A-B path in G, so suppose that $P = (w_0, \ldots, w_\ell)$ is a minimal A-B path with $w_0 \in A$ and $w_\ell \in B$. Note that $\ell \geq 1$ since A and B are assumed to be disjoint and that problem 3 tells us all vertices of P other than w_0 and w_ℓ belong to neither A nor B. Therefore, supposing that $v_i = w_0$ and $u_j = w_\ell$, we see that the following are paths in G:

$$P_1 = (v_0, \ldots, v_{i-1}, v_i = w_0, w_1, \ldots, w_{\ell-1}, w_\ell = u_j, u_{j+1}, \ldots, u_k),$$

$$P_2 = (v_k, \ldots, v_{i+1}, v_i = w_0, w_1, \ldots, w_{\ell-1}, w_\ell = u_j, u_{j-1}, \ldots, u_0).$$

Observe that the length of P_1 is $i + \ell + (k - j) \geq k + 1 + (i - j)$ where we used the fact that $\ell \geq 1$. Similarly, the length of P_2 is $(k - i) + \ell + j \geq k + 1 + (j - i)$. Since either $i \geq j$ or $j \geq i$, we see that one of P_1, P_2 has length at least $k + 1$; contradicting the fact that the length of a maximum path in G is k. □

Problem 5 (2 pts)**.** Prove that a graph G is bipartite if and only if every subgraph H of G has an independent set consisting of at least half of $V(H)$. (recall that an independent set is a set of vertices which induce no edges)

Solution. Suppose first that G is bipartite and let $V(G) = A \sqcup B$ be a bipartition of G. If H is any subgraph of G, then $(V(H) \cap A) \sqcup (V(H) \cap B)$ is a bipartition of H. Therefore both $V(H) \cap A$ and $V(H) \cap B$ are independent sets of H, and at least one must have size at least $|V(H)|/2$.

For the other direction, suppose that G is not bipartite, so G contains an odd cycle (Theorem 1.12). Let C denote this odd cycle (which is a subgraph of G) and suppose it has length k. We showed in class that the largest independent set of C has $(k-1)/2 < |V(C)|/2$ many vertices. □