These solutions are from https://mathematicaster.org/teaching/graphs2022/sol-hw10.pdf

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (2pts)**.** Let G be a graph on $n \geq 2$ vertices with $\delta(G) \geq n/2$. Show that:

1. If n is even, then G has a perfect matching.
2. If n is odd, then $G - v$ has a perfect matching for every $v \in V(G)$.

Solution. Consider first the case of $n = 2$; since $\delta(G) \geq 1$ here, we know that $G \cong K_2$, which has a perfect matching.

Now consider $n \geq 3$. Since $\delta(G) \geq n/2$, Dirac's theorem tells us that G contains a Hamiltonian cycle $(v_1, \ldots, v_n)$.

1. Suppose that n is even. Then the edges $v_1v_2,\ v_3v_4,\ \ldots,\ v_{n-1}v_n$ are vertex-disjoint and cover all vertices of G. In other words, G has a perfect matching.
2. Suppose that n is odd and fix any $v \in V(G)$. Since $(v_1, \ldots, v_n)$ forms a Hamiltonian cycle of G, we may suppose, without loss of generality, that $v_n = v$. Then the edges $v_1v_2,\ v_3v_4,\ \ldots,\ v_{n-2}v_{n-1}$ are vertex-disjoint and cover all vertices of G except for $v_n = v$. In other words, this is a perfect matching in $G - v$.

□

Problem 2 (2pts)**.** Let G be a k-regular bipartite graph. Prove that we can partition the edges of G into k perfect matchings. That is, show that we can can partition $E(G) = M_1 \sqcup \cdots \sqcup M_k$ where each M_i is a perfect matching in G.

Solution. We prove the claim by induction on k.

As a base-case, consider $k = 0$. Then G has no edges and so $E(G) = \varnothing$ can trivially be partitioned into $k = 0$ many perfect matchings.

Now suppose that $k \geq 1$. Then we know that G contains a perfect matching; call one of the M_k. Consider the graph $G' = G - M_k$ (deleting edges here). Since M_k is a perfect matching of G, every vertex of G is incident to exactly one edge in M_k, so G' is $(k-1)$-regular. Thus, by the induction hypothesis, we can partition $E(G') = M_1 \sqcup \cdots \sqcup M_{k-1}$ where $M_1, \ldots, M_{k-1}$ are perfect matchings in G'. Since G' is a spanning subgraph of G, these are also perfect matchings in G. Therefore, $E(G) = E(G') \sqcup M_k = M_1 \sqcup \cdots \sqcup M_k$ as desired. □

Problem 3 (2pts)**.** Let G be a bipartite graph with parts A, B. Prove that if

$$\sum_{a \in A} \frac{1}{\deg a} \leq 1,$$

then G contains a matching which saturates A. Here, we employ the convention that $\frac{1}{0} = +\infty$.

Hint: Recall the following two "tricks" used in class for a somewhat similar problem:

- Silly sizes: If X is a non-empty finite set, then $|X| = \sum_{x\in X} 1$ and $1 = \sum_{x\in X} \frac{1}{|X|}$.
- Switching the order of summation: Fix finite sets X, Y and suppose that $\Omega \subseteq X \times Y$. For any function $f\colon X \times Y \to \mathbb{R}$,

$$\sum_{(x,y)\in\Omega} f(x,y) = \sum_{x\in X} \sum_{\substack{y\in Y:\\(x,y)\in\Omega}} f(x,y) = \sum_{y\in Y} \sum_{\substack{x\in X:\\(x,y)\in\Omega}} f(x,y).$$

Solution. [#1] Since $\frac{1}{0} = +\infty$ and $\sum_{a\in A} \frac{1}{\deg a} \le 1$, we may certainly suppose that $\deg a \ge 1$ for all $a \in A$.

We verify Hall's condition, so fix any non-empty $S \subseteq A$; we must show that $|N(S)| \ge |S|$, which then guarantees the desired matching. To do so, we imitate the main idea in our proof of Theorem 3 on 03-31.

$$\begin{aligned}|S| = \sum_{a\in S} 1 = \sum_{a\in S} \sum_{b\in N(a)} \frac{1}{\deg a} &= \sum_{b\in N(S)} \sum_{a\in N(b)} \frac{1}{\deg a}\\ &\le \sum_{b\in N(S)} \sum_{a\in A} \frac{1}{\deg a} \le \sum_{b\in N(S)} 1 = |N(S)|,\end{aligned}$$

where the first inequality follows from the fact that $N(b) \subseteq A$ for all $b \in B$ and $1/\deg a \ge 0$ for all $a \in A$ and the second inequality follows from the assumption. □

Solution. [#2] Since $\frac{1}{0} = +\infty$ and $\sum_{a\in A} \frac{1}{\deg a} \le 1$, we may certainly suppose that $\deg a \ge 1$ for all $a \in A$.

We verify Hall's condition, so fix any non-empty $S \subseteq A$; we must show that $|N(S)| \ge |S|$. Since $N(a) \subseteq N(S)$ for all $a \in S$, we know that $\deg a \le |N(S)|$ for all $a \in S$. Therefore,

$$1 \ge \sum_{a\in A} \frac{1}{\deg a} \ge \sum_{a\in S} \frac{1}{\deg a} \ge \sum_{a\in S} \frac{1}{|N(S)|} = \frac{|S|}{|N(S)|} \implies |N(S)| \ge |S|.$$

□

Problem 4 (2pts)**.** Fix any non-negative integers n, k such that $k \le (n-1)/2$. Prove that there exists an injection

$$f\colon \binom{[n]}{k} \to \binom{[n]}{k+1}$$

with the property that $X \subseteq f(X)$ for all $X \in \binom{[n]}{k}$.

Note: you do not need to actually define the injection; it is enough to simply show that it exists.

Solution. Consider the bipartite graph G with parts $A = \binom{[n]}{k}$ and $B = \binom{[n]}{k+1}$ where, for $a \in A, b \in B$, we have $ab \in E(G)$ iff $a \subseteq b$. Observe that the desired injection exists if (and only if) G has a matching which saturates A. Indeed, if G has such a matching M, then we can define $f(a) = b$ where $ab \in M$.

For any $a \in A$, we have $\deg a = n - |a| = n - k$ (we build a superset of a of size $k+1$ by appending any non-element of a). For any $b \in B$, we have $\deg b = |b| = k+1$ (we build a subset of b of size k by deleting any element of b).

Now, since $k \leq (n-1)/2$, we find that

$$n - k \geq n - \frac{n-1}{2} = \frac{n-1}{2} + 1 \geq k + 1.$$

In particular, $\deg a = n - k \geq k + 1$ for all $a \in A$ and $\deg b \leq k + 1$ for all $b \in B$. Since $k + 1 \geq 1$, we may thus apply Corollary 4 from 03-31 to find that G has a matching which saturates A, which yields the claim. □

Problem 5 (2pts)**.** Let G be a bipartite graph with parts A, B wherein no vertex of A is isolated. Show that if all vertices in A have distinct degrees, then G contains a matching which saturates A.

Hint: If $S \subseteq [n]$ is non-empty, how do $|S|$ and $\max S$ (the largest element in S) compare?

Solution. [#1] We verify Hall's condition, so fix any non-empty $S \subseteq A$; we must show that $|N(S)| \geq |S|$. Set $k = |S|$. We claim that there is some $a \in S$ with $\deg a \geq k$. Indeed, since $\deg a \geq 1$ for all $a \in A$, if $\deg a \leq k - 1$ for all $a \in S$, then $\deg a \in [k-1]$ for all $a \in S$. But then there would be two vertices in S with the same degree since $k - 1 < k = |S|$; a contradiction.

Thus, since $N(S) \supseteq N(a)$ for each $a \in S$, we have $|N(S)| \geq k = |S|$, which concludes the proof. □

Solution. [#2] We start with a claim:

Claim 1. *Let G be a bipartite graph with parts A, B. If we can label $A = \{a_1, \ldots, a_n\}$ so that $\deg a_i \geq i$ for all $i \in [n]$, then G contains a matching which saturates A.*

Proof. We prove the claim by induction on n. If $n = 1$, then $A = \{a_1\}$ and $\deg a_1 \geq 1$, so there is certainly a matching which saturates A.

Now suppose that $n \geq 2$. Since $\deg a_1 \geq 1$, there is some $b_1 \in B$ with $a_1 b_1 \in E(G)$. Now, form the graph G' by deleting a_1 and b_1, so G' has parts $A' = A \setminus \{a_1\}$ and $B' = B \setminus \{b_1\}$. Note that $\deg_{G'} a \geq \deg_G a - 1$ for all $a \in A'$ since we deleted only one vertex from B. Thus, for all $i \in \{2, \ldots, n\}$, we have $\deg_{G'} a_i \geq \deg_G a_i - 1 \geq i - 1$. So, we can label $A' = \{a'_1, \ldots, a'_{n-1}\}$ where $a'_i = a_{i+1}$ to have $\deg_{G'} a'_i \geq i$. The induction hypothesis allows us to find a matching M in G' which saturates A'. Since neither a_1 nor b_1 exist in G', we find that $M \cup \{a_1 b_1\}$ is a matching in G which saturates A. □

Now for the problem at hand. Since no vertex of A is isolated and all vertices have distinct degrees, we may label $A = \{a_1, \ldots, a_n\}$ so that $1 \leq \deg a_1 < \deg a_2 < \cdots < \deg a_n$. Degrees are integers and so this implies that $\deg a_i \geq i$ for all $i \in [n]$. Thus, G has a matching which saturates A thanks to the above claim. □

Problem 6 (1 bonus point)**.** Fix positive integers m, n and let X be a set of size mn. Also, fix any two partitions $X = A_1 \sqcup \cdots \sqcup A_n$ and $X = B_1 \sqcup \cdots \sqcup B_n$ where $|A_i| = |B_i| = m$ for all $i \in [n]$. Prove that there exists a bijection $\pi \colon [n] \to [n]$ (i.e. a permutation on $[n]$) such that $A_i \cap B_{\pi(i)} \neq \varnothing$ for all $i \in [n]$.

This problem will be graded all-or-nothing.

(Also, I won't lie to you: notation gets really annoying here; as long as your notation is well explained, you'll be fine, even if your notation is somewhat ambiguous.)

Solution. We begin with a lemma.

Lemma 2. *Fix any subset $Y \subseteq X$ and set $I = \{i \in [n] : B_i \cap Y \neq \varnothing\}$. Then $|I| \geq |Y|/m$.*

Proof. We claim first that $\bigcup_{i\in I} B_i \supseteq Y$. Indeed, fix any $y \in Y$. Since $Y \subseteq X = B_1 \sqcup \cdots \sqcup B_n$, we know that $y \in B_i$ for some $i \in [n]$. Furthermore, such an i must have $i \in I$ since $Y \cap B_i \supseteq \{y\}$, which is non-empty. Since $y \in Y$ was arbitrary, we know that $\bigcup_{i\in I} B_i \supseteq Y$. Now, since $|B_i| = m$ for each i, we have

$$|Y| \leq \left|\bigcup_{i\in I} B_i\right| \leq^{1} \sum_{i\in I} |B_i| = |I| \cdot m \qquad \Longrightarrow \qquad |I| \geq |Y|/m. \qquad \square$$

From here, we have an immediate corollary.

Corollary 3. *Fix any non-empty $J \subseteq [n]$ and set $I = \left\{i \in [n] : B_i \cap \bigcup_{j\in J} A_j \neq \varnothing\right\}$. Then $|I| \geq |J|$.*

Proof. Since the A_i's are pairwise disjoint, we have

$$\left|\bigcup_{i\in J} A_j\right| = \sum_{j\in J} |A_j| = |J| \cdot m.$$

Thus, Lemma 2 implies that $|I| \geq (|J| \cdot m)/m = |J|$. $\square$

We are now ready for the proof of the actual claim. Build a bipartite graph G with parts $\mathcal{A} = [n]$ and $\mathcal{B} = [n]$ (note: we consider these to be two different copies of $[n]$, so that $\mathcal{A}$ and $\mathcal{B}$ are disjoint (notation is painful sometimes and I don't want to force any category-theory nonsense (co-products) on you to make this notationally apparent)) where $ab \in E(G)$ ($a \in \mathcal{A}$, $b \in \mathcal{B}$) iff $A_a \cap B_b \neq \varnothing$. Observe that the problem is equivalent to finding a perfect matching in the graph G. Since $|\mathcal{A}| = n = |\mathcal{B}|$, it is enough to show that G has a matching which saturates $\mathcal{A}$.

For any $a \in \mathcal{A} = [n]$, we have

$$N(a) = \left\{b \in \mathcal{B} : B_b \cap A_a \neq \varnothing\right\}.$$

Fix any non-empty $J \subseteq \mathcal{A} = [n]$; we seek to show that $|N(J)| \geq |J|$, which will verify Hall's condition and conclude the proof. Using what was just stated and the definition of the union, we thus have

$$N(J) = \bigcup_{a\in J} \left\{b \in \mathcal{B} : B_b \cap A_a \neq \varnothing\right\} = \left\{b \in \mathcal{B} : B_b \cap \bigcup_{a\in J} A_a \neq \varnothing\right\}.$$

Hence, Corollary 3 implies that $|N(J)| \geq |J|$ and so we have verified Hall's condition, which concludes the proof. $\square$

Problem 7 (2 bonus points)**.** Sportsball is a game played between two teams that cannot end in a tie, so one of these two teams wins the game and the other loses, no matter what. We have a sportsball league with $2n$ teams for some integer $n \geq 1$. Over a season of $2n - 1$ days, every team plays every other team at most once. Furthermore, each team plays at most one game per day.

We have $2n - 1$ trophies to hand out, one for each day of the season. Reasonably, a trophy for a particular day must go to one of the winning teams on that day. We seek to hand out as many trophies as possible so that each team gets at most one trophy. We hand out the trophies at the end of the season, so we can take into account the full results of the season before making our decision.

[1] Really, this is an equality since the B_i's are pairwise disjoint.

Let T be the set of all teams. For each team $t \in T$, let $p(t) \subseteq T \setminus \{t\}$ denote the set of teams that team t played over the course of the season. Prove that we can hand out at least

$$\min_{R \subsetneq T} \max_{t \in T \setminus R} |R \cup p(t)|$$

of the trophies.[2]

This problem will be essentially graded all-or-nothing, except you can get **one of the two bonus points for proving the following special case**:

If each team plays every other team *exactly* once over the course of the season (i.e. $p(t) = T \setminus \{t\}$ for all $t \in T$), then we can hand out all $2n-1$ trophies.

Solution. Let T denote the set of teams and let D denote the set of days, so $|T| = 2n$ and $|D| = 2n-1$. We build a bipartite graph G with parts T and D where $td \in E(G)$ ($t \in T, d \in D$) if and only if t was one of the winning teams on day D. Handing out trophies in the desired fashion corresponds to a matching in G, the number of trophies that we can hand out is exactly $\alpha'(G)$.

For just the one bonus point, we need to show that $\alpha'(G) = |D| = 2n-1$ (i.e. G has a matching which saturates D) assuming that each team plays every other team exactly once.

To show, this, we need to show that for any non-empty subset $S \subseteq D$, we have $|N(S)| \geq |S|$. Observe that $N(S)$ is exactly the set of teams that won some game played on a day in S. In particular, setting $L = T \setminus N(S)$, we see that L is the set of all teams that did not win any game they played on any day in S (a priori, some teams may not have played any games on some/all of these days, though it is possible (but not necessary here) to prove that this is not the case). If $|L| \leq 1$, then $|N(S)| \geq |T| - 1 = 2n-1 = |D| \geq |S|$ and we are done; thus we may suppose that $|L| \geq 2$.

Fix any team $t \in L$. For any other team $s \in L \setminus \{t\}$, by assumption, the teams t and s must have played each other on some unique day $d_s \in D$. Since sportsball cannot end in a tie, we cannot have $d_s \in S$ and so $d_s \in D \setminus S$. Furthermore, since t plays at most one game per day, for any $s, s' \in L \setminus \{t\}$, if $d_s = d_{s'}$, then $s = s'$. In particular, the function $s \mapsto d_s$ is an injection from $L \setminus \{t\}$ to $D \setminus S$ and so $|D \setminus S| \geq |L \setminus \{t\}| = |L| - 1$. Since $|D \setminus S| = 2n-1-|S|$ and $|L| - 1 = |T \setminus N(S)| - 1 = 2n-1-|N(S)|$, we have

$$2n-1-|S| \geq 2n-1-|N(S)| \quad \implies \quad |N(S)| \geq |S|,$$

which concludes the proof.

For both bonus points, we employ the extended form of Hall's theorem:

$$\alpha'(G) = 2n-1-\max_{S \subseteq D} \operatorname{defect}(S) = \min_{S \subseteq D} \bigl(2n-1-\operatorname{defect}(S)\bigr).$$

where $\operatorname{defect}(S) = \max\{0, |S| - |N(S)|\}$. We will prove that

$$2n-1-\operatorname{defect}(S) \geq \min_{R \subsetneq T} \max_{t \in T \setminus R} |R \cup p(t)|, \tag{1}$$

[2] There are instances in which strictly more of the trophies can be handed out. For example, if at most one game is played per day and each team plays exactly once, then we can hand out n trophies, yet the stated expression evaluates to 1. I'd be interested to know if you can derive a better lower bound without taking into account the actual results of any game. I'm more than happy to dish out even more bonus points for such a result, though send any such argument to me separately from your homework.

for every $S \subseteq D$, which will yield the claim.

To begin, for any $R \subsetneq T$ and any $t \in T \setminus R$, since we also have $t \notin p(t)$, we find that

$$|R \cup p(t)| \leq |T \setminus \{t\}| = 2n - 1;$$

so (1) holds if $\text{defect}(S) = 0$.

Thus, fix any $S \subseteq D$ such that $\text{defect}(S) \neq 0$; in particular $\text{defect}(S) = |S| - |N(S)| \geq 1$.

Observe that $N(S)$ is exactly the set of teams that won some game played on a day in S. Setting $L = T \setminus N(S)$, we see that L is the set of all teams that either did not play any game on days in S or lost every game they played on days in S. Since we are assuming that $|S| - |N(S)| \geq 1$ here we cannot have $N(S) = T$ and so $|L| \geq 1$.

Build a graph H with vertex-set L where $xy \in E(H)$ if and only if teams x and y played each other on some day in the season. Note that $N_H(x) = p(x) \cap L$ for all $x \in L$. Furthermore, define the function $f \colon E(H) \to D \setminus S$ where $f(xy) = d$ iff x and y played each other on day d. Observe that f is well-defined since, if x and y play each other, then there is a unique day on which this happens and also of these teams must win this game since there are no ties in sportsball (so $f(xy) \notin S$). Furthermore, for any fixed $x \in L$, we observe that f is injective on $N_H(x)$ since x plays at most one game per day. Therefore, $|p(x) \cap L| = \deg_H x \leq |D \setminus S| = 2n - 1 - |S|$ for every $x \in L$.

Recalling that $L = T \setminus N(S)$, fix any $t \in L$, so $t \notin N(S)$ and $p(t) \cap L = p(t) \setminus N(S)$. Thus, we have $2n - 1 - |S| \geq |p(t) \setminus N(S)|$. Since $\text{defect}(S) = |S| - |N(S)|$ here, we then see that

$$2n - 1 - \text{defect}(S) = 2n - 1 - |S| + |N(S)| \geq |p(t) \setminus N(S)| + |N(S)| = |N(S) \cup p(t)|,$$

for all $t \in L = T \setminus N(S)$. We know that $N(S) \subsetneq T$ (since $L \neq \varnothing$), we can finally bound

$$2n - 1 - \text{defect}(S) \geq \max_{t \in T \setminus N(S)} |N(S) \cup p(t)| \geq \min_{R \subsetneq T} \max_{t \in T \setminus R} |R \cup p(t)|,$$

so S satisfies (1) which concludes the proof. □