These solutions are from https://mathematicaster.org/teaching/graphs2022/sol-hw11.pdf

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (2pts)**.** Let G be any graph. Prove that

$$|E(G)| \geq \binom{\chi(G)}{2}.$$

Solution. [#1] Set $t = \chi(G)$ and let H be a t-critical subgraph of G. We know that $\delta(H) \geq t-1$, which also implies that $|V(H)| \geq t$. Applying the handshaking lemma to H, we thus bound

$$|E(G)| \geq |E(H)| = \frac{1}{2}\sum_{v \in V(H)} \deg_H v \geq \frac{1}{2}\sum_{v\in V(H)}(t-1) = \frac{1}{2}|V(H)|(t-1) \geq \frac{t(t-1)}{2} = \binom{t}{2}.$$

□

Solution. [#2] By definition, we can partition $V(G) = A_1 \sqcup \cdots \sqcup A_{\chi(G)}$ where each A_i is an independent set. We claim that, for every $i \neq j \in [\chi(G)]$, there is some edge between A_i and A_j. Indeed, if there were no such edge, then $A_i \cup A_j$ would also be an independent set of G. But then, we could partition $V(G)$ into $\chi(G)-1$ independent sets by merging A_i and A_j; hence contradicting the definition of $\chi(G)$.

Thus, there is an edge between every pair of these independent sets, which yields at least $\binom{\chi(G)}{2}$ many edges of G since these sets are disjoint. □

Problem 2 (1pt)**.** Generalize Theorem 1 from 04-07:

Let $G = (V, E)$ be any graph and consider coloring the edges of G red and blue (formally, we have a function $f\colon E \to \{\text{red}, \text{blue}\}$). Let G_r be the graph formed by the red edges and let G_b be the graph formed by the blue edges (formally, $G_r = \big(V, f^{-1}(\text{red})\big)$ and $G_b = \big(V, f^{-1}(\text{blue})\big)$). Prove that

$$\chi(G_r)\cdot\chi(G_b) \geq \chi(G).$$

Solution. Let $f_r\colon V \to [\chi(G_r)]$ be a proper coloring of G_r and let $f_b\colon V \to [\chi(G_b)]$ be a proper coloring of G_b. Define the coloring

$$f\colon V \to [\chi(G_r)] \times [\chi(G_b)], \quad \text{where} \quad f(x) = \big(f_r(x), f_b(x)\big),$$

which is a $\chi(G_r)\chi(G_b)$-coloring of G. We claim that f is a proper coloring of G, which will imply the claim. Indeed, fix any edge $xy \in E(G)$.

Case 1: xy is red, so $xy \in E(G_r)$. Then $f_r(x) \neq f_r(y)$ since f_r is a proper coloring of G_r. Thus, $f(x) \neq f(y)$.

Case 2: xy is blue, so $xy \in E(G_b)$. Then $f_b(x) \neq f_b(y)$ since f_b is a proper coloring of G_b. Thus $f(x) \neq f(y)$. □

Problem 3 (1pt)**.** For every pair of positive integers m, n, construct a graph G with the following properties:

- G has $m \cdot n$ many vertices, and
- $\chi(G) = m$, and
- $\chi(\overline{G}) = n$.

Solution. Set $G = K_{\underbrace{n, \ldots, n}_{m}}$, which has mn many vertices. Certainly G is m-partite and so $\chi(G) \leq m$. Furthermore, G contains a copy of K_m (picking one vertex from each cluster) and so $\chi(G) = m$.

On the other hand, $\overline{G}$ is the disjoint union of m many copies of K_n. Since $\chi(K_n) = n$, this tells us that $\chi(\overline{G}) = n$. $\square$

Problem 4 (2pts)**.** Prove that G is 3-critical if and only if $G \cong C_{2n+1}$ for some positive integer n.

Solution. ($\Leftarrow$) We already know that $\chi(C_{2n+1}) = 3$. Furthermore, if H is any proper subgraph of C_{2n+1}, then H cannot contain *any* cycle, let alone an odd one (since the only cycle in C_{2n+1} is itself). Thus, H is bipartite and so $\chi(H) \leq 2$. We conclude that C_{2n+1} is 3-critical.

($\Rightarrow$) Since $\chi(G) = 3$, we know that G is not bipartite. Thus, G must contain a copy of C_{2n+1} for some positive integer n; let H denote such a subgraph which is isomorphic to C_{2n+1}. If $G \neq H$, then H is a proper subgraph of G. But since $H \cong C_{2n+1}$, we have $\chi(H) = 3$; contradicting the fact that G is 3-critical. Thus, $G = H \cong C_{2n+1}$ as needed. $\square$

Problem 5 (2pts)**.** Fix any integer $k \geq 1$. Prove that if G is an n-vertex graph wherein

$$\chi\big(G[N(v)]\big) \leq k, \quad \text{for every } v \in V(G),$$

then $\chi(G) \leq \sqrt{2kn}$.

(Hint: Take motivation from our proof that $\chi(G) \leq \sqrt{2n}$ if G is triangle-free. Be warned, though, there are a couple steps which require more care.)

(Hint: If $0 \leq x \leq y$, then $x \leq \sqrt{xy}$.)

Solution. We prove the claim by induction on n.

If $n \leq 2k$, then $\chi(G) \leq n \leq \sqrt{2kn}$ as needed. Thus, suppose that $n \geq 2k+1$.

If $\Delta(G) \leq \sqrt{2kn} - 1$, then $\chi(G) \leq \Delta(G) + 1 \leq \sqrt{2kn}$ and we are done, so we may suppose that $\Delta(G) > \sqrt{2kn} - 1$. Fix any $v \in V(G)$ with $\deg v = \Delta(G) > \sqrt{2kn} - 1$.

If $V(G) = \{v\} \sqcup N(v)$, then we claim that $\chi(G) \leq 1 + k$. Indeed, $\chi\big(G[N(v)]\big) \leq k$ by assumption so we can find a k-coloring $f \colon N(v) \to [k]$ which is a proper coloring of $G[N(v)]$. Then we may give v color $k+1$ to form a proper $(k+1)$-coloring of G, so $\chi(G) \leq 1 + k$. Now, since $k \geq 1$ and $n \geq 2k$, we bound $\chi(G) \leq 1 + k \leq 2k \leq \sqrt{2kn}$ as needed.

Thus, suppose that $V(G) \neq \{v\} \sqcup N(v)$ and set $H = G - (\{v\} \sqcup N(v))$. We claim that $\chi(G) \leq k + \chi(H)$.

Indeed, let $A_1, \ldots, A_{\chi(H)}$ be a partition of $V(H)$ into $\chi(H)$ many independent sets in H; $\chi(H) \geq 1$ since H has some vertices. Since H is an induced subgraph of G, each A_i is also an independent set in G. Now, v has no neighbors in H, so $A_1 \cup \{v\}$ is also an independent set. Next, $\chi(G[N(v)]) \leq k$ and so we can partition $N(v)$ into $B_1, \ldots, B_k$ where each B_i is an independent set. All together, $A_1 \cup \{v\}, A_2, \ldots, A_{\chi(H)}, B_1, \ldots, B_k$ is a partition of $V(G)$ into $\chi(H) + k$ many independent sets and so $\chi(G) \leq k + \chi(H)$ as claimed.

Now, certainly H also satisfies the hypotheses of the theorem since chromatic numbers can only decrease under taking subgraphs. Thus, since $|V(H)| = n - \deg v - 1 < n$, the induction hypothesis tells us that

$$\chi(H) \leq \sqrt{2k(n - \deg v - 1)} < \sqrt{2k(n - \sqrt{2kn})},$$

since $\deg v > \sqrt{2kn} - 1$. Therefore, since $n \geq 2k$,

$$\begin{aligned}\chi(G) &\leq k + \chi(H) < k + \sqrt{2k(n - \sqrt{2kn})} < k + \sqrt{2kn - 2k\sqrt{2kn} + k^2} \\ &= k + \sqrt{(\sqrt{2kn} - k)^2} = \sqrt{2kn}.\end{aligned}$$

□

Problem 6 (2pts)**.** For graphs G, H, the graph G is said to be H-free if G does not contain a copy of H.

Fix any integer $k \geq 3$. Prove that if G is a C_k-free graph on n vertices, then $\chi(G) \leq \sqrt{2(k-2)n}$. (Hint: Problem 5)

Solution. Fix any $v \in V(G)$, thanks to Problem 5, it suffices to prove that $\chi(G[N(v)]) \leq k - 2$. Suppose for the sake of contradiction that $\chi(G[N(v)]) \geq k - 1$. Then, thanks to Theorem 2 from 04-12, we know that $G[N(v)]$ contains a copy of every tree on $k - 1$ many vertices. In particular, it contains a path on $k - 1 \geq 2$ many vertices, label such a path $(v_1, \ldots, v_{k-1})$ where $v_i \in N(v)$ for all $i \in [k-1]$. Since v is adjacent to all of the v_i's, it is, in particular, adjacent to v_1 and v_{k-1}. But then, since $v \notin N(v)$, we find that $(v_1, \ldots, v_{k-1}, v)$ forms a cycle of length k in G; a contradiction since G is assumed to be C_k-free. □