These solutions are from https://mathematicaster.org/teaching/graphs2022/sol-hw12.pdf

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (2 + 2 + 2 pts)**.**

1. Let G be a bipartite graph with parts A, B where $|A| = |B| = n \geq 1$. Prove that if $|E(G)| > n(n-1)$, then G has a perfect matching.

   (It may be easier to rely on Kőnig here instead of on Hall, but it's up to you. You could even just induct on n.)

2. Let G be a bipartite graph with parts A, B and fix an integer $n \geq 1$. Let $A = A_1 \sqcup \cdots \sqcup A_n$ and $B = B_1 \sqcup \cdots \sqcup B_n$ be any partitions (some of the A_i's or B_j's may be empty). Note that $|A|$ and $|B|$ have nothing to do with n; n is just the number of pieces in each partition.

   Prove that if $|E(G)| < n$, then there is a bijection $\pi\colon [n] \to [n]$ such that G has no edges between A_i and $B_{\pi(i)}$ for each $i \in [n]$.

   You are free to use part 1 as a black-box even if you haven't proved it.

3. For each positive integer t, prove that if G is a t-critical graph, then $\lambda(G) \geq t - 1$.

   (Hint: Consult the notes from 03-01. Use part 2 to "merge independent sets".)

   You are free to use parts 1 and/or 2 as a black-box even if you haven't proved then.

   N.b. Vertex-connectivity is a very different story... In particular, the obvious analogue for vertex-connectivity is false (in general). The "Moser spindle" is a counter-example when $t = 4$, and there are many, many others.

Solution.

1. **Via Kőnig:**

   First, since G is bipartite and each side has size n, we know that $\deg v \leq n$ for each $v \in V(G)$.

   Let $C \subseteq V(G)$ be a minimum vertex-cover of G, so $|C| = \beta(G)$. Now, each vertex $c \in C$ covers $\deg c$ many edges of G, so, since C covers every edge of G, we must have

   $$n(n-1) < |E(G)| \leq \sum_{c \in C} \deg c \leq \sum_{c \in C} n = n|C| \implies \beta(G) = |C| > n - 1.$$

   Since $\beta(G)$ and $n-1$ are integers, this means that $\beta(G) \geq n$. Thus, Kőnig tells us that $\alpha'(G) = \beta(G) \geq n$ and so G has a perfect matching.

   Via Hall:

   Since $|A| = |B|$, we just need to show that G has a matching which saturates A. Fix any non-empty $S \subseteq A$; we must show that $|N(S)| \geq |S|$. Set $B' = B \setminus N(S)$. By definition, G has no edges between S and B', and so

   $$n(n-1) < |E(G)| \leq n^2 - |S| \cdot |B'| \implies |S| \cdot |B'| < n \implies |S| \cdot (n - |N(S)|) < n.$$

Of course, $|S| \cdot (n - |N(S)|)$ and n are both integers, so $|S| \cdot (n - |N(S)|) \leq n - 1$. If $|S| = k$, then, since certainly $k \leq n$, we have

$$n - |N(S)| \leq \frac{n-1}{k} \implies |N(S)| \geq n - \frac{n-1}{k} = \frac{(k-1)n+1}{k} \geq \frac{(k-1)k+1}{k} = k - 1 + \frac{1}{k} > k - 1.$$

Again, $|N(S)|$ and $k - 1$ are integers, so, in fact, $|N(S)| \geq k = |S|$ as needed.

Via induction:

If $n = 1$ then the claim is clear since $|E(G)| > 0 \implies |E(G)| = 1$, so this edge is a perfect matching. Thus suppose that $n \geq 2$.

We begin by claiming that there is some $a \in A$ with $\deg a = n$, so this a is adjacent to every vertex in B. Indeed, if there is no such $a \in A$, then $\deg a \leq n - 1$ for all $a \in A$ and so the bipartite handshaking lemma tells us that

$$n(n-1) < |E(G)| = \sum_{a \in A} \deg a \leq \sum_{a \in A} (n-1) = n(n-1);$$

a contradiction.

Now we consider B.

Case 1: Every $b \in B$ has $\deg b = n$. Then $G \cong K_{n,n}$ which we know has a perfect matching.

Case 2: There is some $b \in B$ with $\deg b \leq n - 1$.

Label $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ so that $\deg a_n = n$ and $\deg b_n \leq n - 1$. Note that $a_n b_n \in E(G)$. Set $A' = \{a_1, \ldots, a_{n-1}\}$ and $B' = \{b_1, \ldots, b_{n-1}\}$ and define G' to be the subgraph of G induced on $A' \sqcup B'$, so both parts of G' have size $n-1$. Now, since $a_n b_n \in E(G)$, we have

$$|E(G')| = |E(G)| - (\deg a_n + \deg b_n - 1) > n(n-1) - (n + (n-1) - 1) = (n-1)(n-2).$$

Thus, the induction hypothesis tells us that G' contains a perfect matching M. By definition, neither a_n nor b_n is an end-point of an edge in M, so, since $a_n b_n \in E(G)$, we find that $M \cup \{a_n b_n\}$ is a perfect matching in G.

2. Build a bipartite graph H with parts $\mathcal{A} = \{A_1, \ldots, A_n\}$ and $\mathcal{B} = \{B_1, \ldots, B_n\}$ where $A_i B_j \in E(H)$ iff there are no edges of G between A_i and B_j. Then the desired bijection π exists if and only if H contains a perfect matching.

   Now, the A_i's and B_j's are disjoint, so for each $e \in E(G)$, the edge e connects exactly one pair A_i and B_j. Some of these edges may connect the same A_i and B_j, but, in any case, there are at most $|E(G)| < n$ many *non*-edges in H between $\mathcal{A}$ and $\mathcal{B}$. Thus,

$$|E(H)| > n^2 - n = n(n-1),$$

   and so H has a perfect matching thanks to part 1.

3. $\lambda(G) \geq 0$ always, and so we are done if $t = 1$; thus we may suppose that $t \geq 2$. In this case, G must have at least two vertices.

   Suppose for the sake of contradiction that $\lambda(G) \leq t - 2$. Then we can partition $V(G) = A \sqcup B$ such that A, B are non-empty and $|E[A, B]| \leq t - 2$. Since A and B are non-empty, both

$G[A]$ and $G[B]$ are proper subgraphs of G and so each has chromatic number at most $t-1$ since G is t-critical. Thus, we may partition $A = A_1 \sqcup \cdots \sqcup A_{t-1}$ and $B = B_1 \sqcup \cdots \sqcup B_{t-1}$ so that each A_i is an independent set in $G[A]$ and each B_j is an independent set in $G[B]$. Since $G[A]$ and $G[B]$ are induced subgraphs of G, we know that each A_i and each B_j is also an independent set in G. Now, by considering the bipartite subgraph of G with parts A, B and edges $E[A, B]$, since $|E[A, B]| \le t-2 < t-1$, part 2 hands us a bijection $\pi\colon [t-1] \to [t-1]$ such that there are no edges between A_i and $B_{\pi(i)}$. In particular, for each $i \in [n]$, we know that $A_i \sqcup B_{\pi(i)}$ is an independent set in G. Thus, since π is a bijection, we can write

$$V(G) = A \sqcup B = \big(A_1 \sqcup \cdots \sqcup A_{t-1}\big) \sqcup \big(B_1 \sqcup \cdots \sqcup B_{t-1}\big) = \bigsqcup_{i=1}^{t-1} \big(A_i \sqcup B_{\pi(i)}\big),$$

which yields a partition of $V(G)$ into $t-1$ many independent sets. This, however, implies that $\chi(G) \le t-1$, which contradicts the assumption that G is t-critical.

□

Problem 2 (2pts)**.** Let $g \ge 2$ be an integer and let G be a connected plane graph on n vertices wherein every face is bounded by a cycle of G. Prove that if G has no cycles of length g or smaller, then

$$|E(G)| \le \frac{g+1}{g-1}(n-2).$$

Solution. Since each face of G is bounded by a cycle and G has no cycles of length $\le g$, we must have $\operatorname{len}(f) \ge g+1$ for all $f \in F(G)$. Thus, the headshaking lemma yields

$$2|E(G)| = \sum_{f \in F(G)} \operatorname{len}(f) \ge \sum_{f \in F(G)} (g+1) = (g+1)|F(G)| \implies |F(G)| \le \frac{2}{g+1}|E(G)|.$$

Now, G is connected and so Euler's formula tells us that

$$2 = n + |F(G)| - |E(G)| \le n + \frac{2}{g+1}|E(G)| - |E(G)| = n - \frac{g-1}{g+1}|E(G)| \implies |E(G)| \le \frac{g+1}{g-1}(n-2).$$

□

Problem 3 (2pts)**.** Prove a special case of the 4-color theorem: If G is a planar, triangle-free graph, then $\chi(G) \le 4$.

Solution. [#1] Suppose for the sake of contradiction that $\chi(G) \ge 5$ and let H be any 5-critical subgraph of G. Since H is a subgraph of G, H is also planar and triangle-free. Additionally, H is connected and has $\delta(H) \ge 4$ (Props 4&6 from 04-12). Set $n = |V(H)|$; certainly $n \ge 5 \ge 3$ since $\delta(H) \ge 4$.

We may therefore apply the handshaking lemma and Theorem 11 from 04-14 to bound

$$2n - 4 \ge |E(H)| = \frac{1}{2} \sum_{v \in V(H)} \deg_H v \ge \frac{1}{2} \sum_{v \in V(H)} 4 \ge \frac{4n}{2} = 2n;$$

a contradiction.

□

Solution. [#2] Let G be a planar, triangle-free graph; we claim that $\delta(G) \le 3$. If G has connected components $G_1, \dots, G_k$, then $\delta(G) = \min_{i \in [k]} \delta(G_i)$, so it suffices to consider the case when G is connected. Set $n = |V(G)|$. If $n \le 2$, then $\delta(G) \le 1$, so we may suppose that $n \ge 3$. So G is a planar, triangle-free, connected graph with $n \ge 3$, so we may apply the handhsaking lemma and Theorem 11 from 04-14 to bound

$$2n - 4 \ge |E(G)| = \frac{1}{2} \sum_{v \in V(G)} \deg v \ge \frac{1}{2} \sum_{v \in V(G)} \delta(G) = \frac{n}{2}\delta(G) \implies \delta(G) \le \frac{2}{n}(2n-4) = 4 - \frac{8}{n} < 4.$$

Since $\delta(G)$ and 4 are integers, we conclude that $\delta(G) \le 3$.

Now, if H is any subgraph of G, then H is also planar and triangle-free. Therefore,

$$d(G) = \max\{\delta(H) : H \text{ is a subgraph of } G\} \le 3,$$

and so $\chi(G) \le d(G) + 1 \le 4$ as needed. □