These solutions are from https://mathematicaster.org/teaching/graphs2022/sol-hw13.pdf

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (2pts)**.** The 3-color Ramsey number $R(m, n, p)$ is the smallest integer N such that every 3-coloring of $E(K_N)$ (say with colors red, blue and green) contains either a red copy of K_m, a blue copy of K_n or a green copy of K_p.

Prove that $R(3, 3, 3) \leq 17$.

N.b. It turns out that $R(3, 3, 3) = 17$, but I'm not asking you to prove the lower bound.

Solution. Fix any red,blue,green-coloring of $E(K_{17})$ and fix any vertex v. Let R, B, G denote the red, blue and green neighborhoods of v, respectively, so $N(v) = R \sqcup B \sqcup G$. In particular, $|R|+|B|+|G| = 16$. If all three of these sets had size at most 5, then we would have $|R|+|B|+|G| \leq 15$; a contradiction. Thus, one of these neighborhoods must have size at least 6. Since we are looking for the same structure in each color, without loss of generality, we may suppose that $|R| \geq 6$.

Now, consider restricting the coloring to $\binom{R}{2}$. If there is any edge $xy \in \binom{R}{2}$ which is colored red, then $\{v, x, y\}$ would induce a red triangle and we would be done! Otherwise, every edge in $\binom{R}{2}$ is either blue or green. Then, since $|R| \geq 6 = R(3, 3)$, we know that this restricted coloring contains either a blue triangle or a green triangle as needed. □

Problem 2 (2pts)**.** Prove that $R(3, n) \leq n^2$ for every positive integer n.

(Hint: It's probably best if you just think about G vs $\overline{G}$ instead of red,blue-colorings here.)

(Hint: Review the *silly* upper and lower bounds that we proved on chromatic numbers on 04-05.)

Solution. [#1] Let G be any n^2-vertex graph; we must show that either G contains a triangle or that $\overline{G}$ contains a copy of K_n (i.e. $\alpha(G) \geq n$)

Thus, suppose that G does not contain a triangle; we must show that $\alpha(G) \geq n$.

To begin, fix any $v \in V(G)$. Since G is triangle-free, it must be the case that $N(v)$ is an independent set in G. Therefore, if $\deg v = |N(v)| \geq n$, then $\alpha(G) \geq n$ and we are done. Since this is true for every $v \in V(G)$, we may suppose that $\Delta(G) \leq n - 1$.

However, in this case we can combine our silly bounds on $\chi(G)$ to find that

$$\frac{n^2}{\alpha(G)} \leq \chi(G) \leq \Delta(G) + 1 \leq n \quad \implies \quad \alpha(G) \geq \frac{n^2}{n} = n,$$

as needed. □

Solution. [#2] I was dumb and didn't realize that an even stronger bound follows from the basic bounds we derived in class... Oh well, free points for you!

Since $n \geq 1$, we have $n + 1 \leq 2n$, so

$$R(3, n) \leq \binom{n + 3 - 2}{3 - 1} = \frac{n(n + 1)}{2} \leq n^2.$$

□

Problem 3 (2pts)**.** Consider coloring the numbers in [17] red and blue, i.e. we have a function $f\colon [17] \to \{\text{red}, \text{blue}\}$. Prove that there exist $a \neq b \in [17]$ such that $f(a) = f(b) = f(a+b)$. (Note that no integers outside of [17] receive any color, so it must be the case that also $a + b \in [17]$ in order for this to happen.)

(Hint: Create a red,blue-coloring of $E(K_{18})$ (thinking of $V(K_{18}) = [18]$) based on the differences between the points.)

(Hint: $(x - y) + (z - x) = z - y$.)

You can receive 1.25pts/2 if you prove a simplified version of the claim where you allow $a = b$. (such a proof could completely ignore the hint if you so choose and can also be accomplished when 17 replaced by 5, but you don't have to)

I'm not schur if 17 is the smallest integer for which the claim is true. I'll hand out **1 bonus point** if you can determine (with proof) the smallest integer for which the claim is still true.

Solution. As suggested, for $xy \in E(K_{18})$, set $g(xy) = f(|x - y|)$. Since $x \neq y \in [18]$, we know that $|x-y| \in [17]$ and so g is well-defined. Since g is a red,blue-coloring of $E(K_{18})$ and $R(4,4) = 18$, we know that g admits a monochromatic copy of K_4 since $R(4,4) \leq 18$; label the vertices of such a monochromatic copy of K_4 as $x < y < z < w$.

Begin by setting $a = y - x$ and $b = z - y$, so $b + a = z - x$. By definition $f(a) = f(b) = f(a+b)$, so if $a \neq b$, then we're done. Thus, suppose that $a = b$, so $y - x = z - y$.

Next, set $a = y - x$ and $b = w - y$, so $a + b = w - x$. Again, by definition $f(a) = f(b) = f(a+b)$, so if $a \neq b$, then we're done. If, however, $a = b$, then, using what we learned in the previous paragraph, we would have $w - y = y - x = z - y \implies w - y = z - y \implies w = z$; a contradiction since we know that $z < w$.

For the simplified version of the claim, we don't really need Ramsey at all[1] and can replace 17 by 5. Indeed, let $f\colon [5] \to \{\text{red}, \text{blue}\}$ be any red,blue-coloring. Suppose for the sake of contradiction that there are no $a, b \in [5]$ with $f(a) = f(b) = f(a + b)$. Since the colors are symmetric, we may, without loss of generality, suppose that $f(1) = \text{red}$. Now, since we allow $a = b$, we find that $f(1) \neq f(2)$ since $1 + 1 = 2$ and so $f(2) = \text{blue}$. Also, $2 + 2 = 4$, so $f(2) \neq f(4)$ implying that $f(4) = \text{red}$. Then, since $1 + 4 = 5$ and $f(1) = f(4) = \text{red}$, so we must have $f(5) = \text{blue}$. Also, since $1 + 3 = 4$ and $f(1) = f(4) = \text{red}$, we mus also have $f(3) = \text{blue}$. But then $2 + 3 = 5$ and $f(2) = f(3) = f(5) = \text{blue}$; a contradiction. □

Problem 4 (2 + 2 pts)**.** Let n be a positive integer and set $N = 3n - 1$.

1. Prove that every red,blue-coloring of $E(K_N)$ contains a monochromatic matching with n edges.[2]

   (Hint: Show that there exists an incident red- and blue-edge unless every edge gets the same color)

[1] Granted, you could probably brute-force your way through a proof of the full claim without relying in Ramsey at all, but that sounds very, very painful.

[2] Fun fact (if you care). Suppose that the red,blue-coloring of $E(K_N)$ was initially hidden from you and, one by one, you're allowed to ask about the color of whichever edges you desire. How many edges do you have to ask about in order to actually find a monochromatic matching with n edges? Of course, you can find the matching by asking about about the color of all $\binom{N}{2}$ edges thanks to this problem; however, you can do much, much better. It turns out that you can actually get away with asking about only $N - 1$ of the edges! The general case (more than two colors) is still open, though, if you care to think about it :) The relevant papers are https://arxiv.org/abs/1904.00246 and https://arxiv.org/abs/2010.04113.

2. Construct a red,blue-coloring of $E(K_{N-1})$ which *does not* contain a monochromatic matching with n edges.

   (Hint: Break $V(K_{N-1})$ into two pieces, one of size $2n-1$ and the other of size $n-1$, and color the edges based on how they intersect these two pieces)

Solution.

1. We prove the claim by induction on n.

   If $n = 1$, then $N = 2$. Of course any red,blue-coloring of $E(K_2)$ contains a monochromatic matching with at least 1 edge since the only edge in existence gets a color!

   Thus, suppose that $n \geq 2$ and fix any red,blue-coloring of $E(K_N)$.

   Since $N \geq 2$, we may suppose that the color red is used in the coloring (since we could switch the colors otherwise given that the colors are symmetric here). If every edge is red, then there is certainly a red matching with at least n edges since $N = 3n - 1 \geq 2n$.

   Now, suppose there is at least one red edge and at least one blue edge; suppose that xy is red and uv is blue. If $\{x, y\} \cap \{u, v\} \neq \varnothing$, then these edges are incident. Otherwise, consider the edge yu. If yu is blue, then xy and yu are incident red and blue edges. Otherwise, yu is red, so yu and uv are incident red and blue edges.

   In either case, there is some vertex v which is incident to some red edge and to some blue edge. Suppose that vx is red and vy is blue for some x, y. Set $V' = V(K_N) \setminus \{v, x, y\}$, so $|V'| = N - 3 = 3n - 1 - 3 = 3(n-1) - 1$. Consider the restricting the coloring to $\binom{V'}{2}$. By the induction hypothesis, there exists a monochromatic matching with $n-1$ edges which lives within $\binom{V'}{2}$; call one such matching M. If M is red, then $M \cup \{vx\}$ is a red matching with n edges. If M is blue, then $M \cup \{vy\}$ is a blue matching with n edges.

2. Since $N = 3n - 1$, we can partition $V(K_{N-1}) = A \sqcup B$ where $|A| = 2n - 1$ and $|B| = n - 1$. Color every edge with both end-points in A red and color every other edge blue. We claim that this coloring has no monochromatic matching with n edges.

   Indeed, suppose for the sake of contradiction that M were a monochromatic matching with n edges.

   If M is red, then every end-point of an edge in M must reside in A. Since M is a matching, all of these end-points are distinct and so we must have $|A| \geq 2n$, which is not the case.

   If M is blue, then every edge in M must have at least one end-point in B (since otherwise that edge would be red). But M has n many edges and these edges are vertex-disjoint, so this would only be possible if $|B| \geq n$, which is not the case.

$\square$