These solutions are from https://mathematicaster.org/teaching/graphs2022/sol-hw2.pdf

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (1 pt)**.** Recall that a *directed graph* (or digraph) is a pair $D = (V, E)$ where V is a set and $E \subseteq V^2$. For $u, v \in V$, a u-v diwalk in D is a sequence $(u = v_0, \dots, v_k = v)$ such that $(v_i, v_{i+1}) \in E$ for all $i \in \{0, \dots, k-1\}$.

Consider the relation R on V where $u\,R\,v$ iff there is a u-v diwalk. Give an example of a digraph D where R is **not** an equivalence relation. Justify your answer. (Feel free to draw a picture of D)

Solution. Consider $D = \big(\{1, 2\}, \{(1, 2)\}\big)$. Observe that $1R2$ since $(1, 2)$ is a 1-2 diwalk. However, $(2, 1) \notin R$ since there is no 2-1 diwalk because $(2, 1) \notin E$. Therefore R fails to be symmetric and so cannot be an equivalence relation. □

Problem 2 (2 pts)**.** Let G be a connected graph and consider a function $f\colon V(G) \to X$ where X is some arbitrary set. Prove that if f is *not* a constant function, then there is an edge $uv \in E(G)$ such that $f(u) \neq f(v)$.

Solution. [#1] Fix any $x \in X$ which is in the image of f; set $A = \{v \in V(G) : f(v) = x\}$ and $B = \{v \in V(G) : f(v) \neq x\}$. Since f is not constant and x was chosen to be in the image of f, we see that A, B are both non-empty and also that $V(G) = A \sqcup B$. Now, since G is connected, there must be an edge $ab \in E(G)$ with $a \in A$ and $b \in B$. This edge has the property that $f(a) = x \neq f(b)$ as needed. □

Solution. [#2] Since f is not constant, there must be some $u, v \in V(G)$ with $f(u) \neq f(v)$. Since G is connected, there is a u-v path; call it $(u = v_0, v_1, \dots, v_k = v)$. Let $i \in \{0, \dots, k\}$ be the largest index for which $f(v_i) = f(u)$. Note that such an i exists since (trivially) $f(v_0) = f(u)$ and also that $i < k$ since $f(v_k) = f(v) \neq f(u)$. Thus, by the definition of i, we know that $v_i v_{i+1} \in E(G)$ and $f(v_i) = f(u) \neq f(v_{i+1})$ as needed. □

Problem 3 (2 pts)**.** Prove that every graph on at least two vertices has a pair of vertices with the same degree.

Solution. Suppose that G has n vertices; thus for any $v \in V(G)$, we have $\deg v \in \{0, \dots, n-1\}$. Suppose for the sake of contradiction that no two vertices have the same degree; thus we may label the vertices $v_0, \dots, v_{n-1}$ so that $\deg v_i = i$. Consider v_{n-1} which has degree $n-1$. This means that v_{n-1} is adjacent to every other vertex of G; in particular, $v_0 v_{n-1} \in E(G)$ since $n \geq 2$ by assumption. However, $\deg v_0 = 0$ and so v_0 has no neighbors; a contradiction. □

Problem 4 (2 pts)**.** Prove that if G is a graph with $\delta(G) \geq 2$, then G must contain a cycle.

Solution. Suppose that $P = (v_0, v_1, \dots, v_k)$ is any maximal path in G. Since $\delta(G) \geq 2$, we know that $\deg v_0 \geq 2$; we can thus find some $u \in V(G) \setminus \{v_1\}$ such that $v_0 u \in E(G)$. We claim that $u \in V(P)$. Indeed, if not, then $(u, v_0, v_1, \dots, v_k)$ is a path in G, which contradicts the maximality of the path P. As such, $u = v_i$ for some $i \in \{2, \dots, k\}$ (since $u \neq v_1$); therefore $(v_0, v_1, \dots, v_i)$ forms a cycle in G. □

Problem 5 (3 pts)**.** Let G be a graph and let A be an independent set of G. Prove that

$$\sum_{v\in A} \deg v \leq |E(G)|$$

with equality if and only if G is bipartite with parts A and $V(G) \setminus A$.
(Especially in this problem, be sure to carefully justify all steps in your argument)

Solution. [#1] To begin, we point out that, since A is already assumed to be an independent set, G is bipartite with parts A and $V(G) \setminus A$ if and only if $V(G) \setminus A$ is an independent set.

Suppose that $G = (V, E)$ and for $v \in A$ define

$$E_v = \{e \in E : v \in e\},$$

i.e. all edges incident to v. Note that $|E_v| = \deg v$.

We claim first that for any $u \neq v \in A$, we must have $E_u \cap E_v = \varnothing$. Indeed, if $e \in E_u \cap E_v$, then $u \in e$ and $v \in e$; since $u \neq v$ this means that $e = uv$. However, A is an independent set and so this is impossible.

Thus, set $\widehat{E} = \bigcup_{v\in A} E_v$. Since the E_v's are pairwise disjoint and certainly $\widehat{E} \subseteq E$, we have

$$|E| \geq |\widehat{E}| = \sum_{v\in A} |E_v| = \sum_{v\in A} \deg v,$$

with equality if and only if $E = \widehat{E}$.

To finish the proof, We need to show that $E = \widehat{E}$ if and only if $V \setminus A$ is an independent set. We already know that $\widehat{E} \subseteq E$, so fix any $e \in E$. Observe that $e \notin \widehat{E}$ if and only if $e \cap A = \varnothing$ which happens if and only if $e \subseteq V \setminus A$. In other words, $e \in \widehat{E}$ for all $e \in E$ if and only if $V \setminus A$ is an independent set; thus the claim follows. □

Solution. [#2] To begin, we point out that, since A is already assumed to be an independent set, G is bipartite with parts A and $V(G) \setminus A$ if and only if $V(G) \setminus A$ is an independent set.

Suppose that $G = (V, E)$ and define the set

$$\widehat{E} = \{(v, e) \in A \times E : v \in e\}.$$

We begin by noticing that

$$|\widehat{E}| = \sum_{v\in A} |\{e \in E : v \in e\}| = \sum_{v\in A} \deg v. \tag{1}$$

Now, fix any $e \in E$ and consider $A_e = \{v \in A : v \in e\}$. Since A is an independent set, we know that $|A_e| \leq 1$ since if $|A_e| = 2$, then both end-points of e would live in A, contradicting the fact that A is an independent set. As such, define

$$E_0 = \{e \in E : |A_e| = 0\}, \quad \text{and} \quad E_1 = \{e \in E : |A_e| = 1\}.$$

By the earlier remark, we know that $E = E_0 \sqcup E_1$. We then compute

$$|\widehat{E}| = \sum_{e\in E} |A_e| = \sum_{e\in E_0} 0 + \sum_{e\in E_1} 1 = |E_1| \leq |E|, \tag{2}$$

with equality if and only if $E_1 = E$ since we already know that $E_1 \subseteq E$. Combining equations (1) and (2) then yields

$$\sum_{v \in A} \deg v \leq |E|,$$

with equality if and only if $E_1 = E$.

To finish the claim, we observe that $E_1 = E$ if and only if $V \setminus A$ is an independent set. Indeed, observe that $e \notin E_1$ if and only if $A_e = \varnothing$, which means that $e \subseteq V \setminus A$. In other words, $e \in E_1$ for all $e \in E$ (and thus $E_1 = E$) if and only if $V \setminus A$ is an independent set. □

Solution. [#3] To begin, we point out that, since A is already assumed to be an independent set, G is bipartite with parts A and $V(G) \setminus A$ if and only if $V(G) \setminus A$ is an independent set.

Suppose that $G = (V, E)$ and define the set

$$\widehat{E} = \{(a, b) \in A \times V : ab \in E\}.$$

We first prove that $|\widehat{E}| \leq |E|$ with equality if and only if $V \setminus A$ is an independent set. To do so, consider the function $f \colon \widehat{E} \to E$ defined by $f(a, b) = \{a, b\}$; note that f is well defined by the definition of $\widehat{E}$. We claim that f is an injection. Indeed, suppose that $f(a_1, b_1) = f(a_2, b_2)$, so $\{a_1, b_1\} = \{a_2, b_2\}$. If $a_1 = a_2$, then $b_1 = b_2$ so $(a_1, b_1) = (a_2, b_2)$ and we are done; thus suppose that $a_1 \neq a_2$. We then have $a_1 = b_2$ and $b_1 = a_2$; thus both $(a_1, b_1) \in \widehat{E}$ and $(b_1, a_1) \in \widehat{E}$. But this means that $a_1, b_1 \in A$ and $a_1 b_1 \in E$ which contradicts the fact that A is an independent set.

Now that we know that f is an injection, we know that $|\widehat{E}| = |E|$ if and only if f is a surjection. Observe that f is a surjection if and only if for every $\{x, y\} \in E$, $(x, y) \in \widehat{E}$ or $(y, x) \in \widehat{E}$, i.e. $x \in A$ or $y \in A$. In other words, f is a surjection if and only if $V \setminus A$ is an independent set.

Now to finish the proof. We compute

$$|E| \geq |\widehat{E}| = \sum_{v \in A} |\{b \in V : vb \in E\}| = \sum_{v \in A} \deg v$$

with equality if and only if $V \setminus A$ is an independent set. □