These solutions are from https://mathematicaster.org/teaching/graphs2022/sol-hw3.pdf

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (2 pts)**.** Let G be a graph on n vertices. Prove that if $\Delta(G) + \delta(G) \geq n - 1$ then G is connected.

Solution. [#1] We prove the contrapositive, so suppose that G is disconnected. We can thus write $G = G_1 \sqcup G_2 \sqcup \cdots \sqcup G_k$ for some $k \geq 2$ where the G_i's are the connected components of G. Set $n_i = |V(G_i)|$; without loss of generality, we may suppose that $n_1 \geq n_2 \geq \cdots \geq n_k$. We now observe that

$$\Delta(G) = \max_{i \in [k]} \Delta(G_i) \leq \max_{i \in [k]} (n_i - 1) = n_1 - 1,$$
$$\delta(G) = \min_{i \in [k]} \delta(G_i) \leq \min_{i \in [k]} (n_i - 1) = n_k - 1.$$

Therefore,

$$\Delta(G) + \delta(G) \leq n_1 - 1 + n_k - 1 \leq n - 2.$$

□

Solution. [#2] If $n = 1$ then G is automatically connected, so we may suppose throughout that $n \geq 2$.

Let $v^* \in V(G)$ be such that $\deg v^* = \Delta(G)$. Observe that it suffices to show that there is a v^*-v walk in G for every $v \in V(G)$ (equivalence relation!). Thus fix any $v \in V(G)$. If $v = v^*$ or if $vv^* \in E(G)$, then we are done, so suppose that neither of these hold.

Next, observe that

$$\deg v^* + \deg v \geq \Delta(G) + \delta(G) \geq n - 1.$$

We showed in class that, since $vv^* \notin E(G)$, this means that there is some $w \in N(v^*) \cap N(v)$; in particular, (v^*, w, v) is a v^*-v walk as needed. □

Problem 2 (2 pts)**.** Let G be a graph with the property that $\deg u + \deg v \equiv 1 \pmod 2$ for every $uv \in E(G)$. Prove that $|E(G)|$ is even.

Solution. For $i \in \{0, 1\}$, let $A_i = \{v \in V(G) : \deg v \equiv i \pmod 2\}$. Since $0 + 0 \equiv 1 + 1 \equiv 0 \pmod 2$, we observe that both A_0 and A_1 are independent sets of G; in particular, G is bipartite with parts A_0 and A_1. We can then apply HW2.5 to compute

$$|E(G)| = \sum_{v \in A_0} \deg v \equiv \sum_{v \in A_0} 0 \pmod 2 \equiv 0 \pmod 2,$$

as claimed. □

Problem 3 (2 pts)**.** Let G be a connected graph and suppose that $f \colon V(G) \to \mathbb{Z}$ is a function with the property that $f(u) + f(v) \equiv 0 \pmod 3$ for every $uv \in E(G)$. Prove that if G is *not* bipartite, then $f(v) \equiv 0 \pmod 3$ for every $v \in V(G)$.

Solution. We prove the contrapositive, so suppose that there is some $v^* \in V(G)$ for which $f(v) \not\equiv 0 \pmod 3$; we need to then show that G is a bipartite graph.

For $i \in \{0,1,2\}$, let $A_i = \{v \in V(G) : f(v) \equiv i \pmod 3\}$. Since $1+1 \equiv 2 \pmod 3$ and $2+2 \equiv 1 \pmod 3$, we observe that A_1 and A_2 are both independent sets of G. Since $A_1 \cap A_2 = \varnothing$, we will have thus succeeded in showing that G is bipartite if we can argue that $A_0 = \varnothing$.

To this end, suppose for the sake of contradiction that $A_0 \neq \varnothing$ and set $B = A_1 \cup A_2$. Note that $B = \{v \in V(G) : f(v) \not\equiv 0 \pmod 3\}$; in particular $B \neq \varnothing$ since $v^* \in B$. Thus $V(G) = A_0 \sqcup B$ and both A_0 and B are nonempty, so since G is connected, there must be some edge $ab \in E(G)$ with $a \in A_0$ and $b \in B$. However, this edge has $f(a) + f(b) \equiv f(b) \pmod 3 \not\equiv 0 \pmod 3$; a contradiction. □

Problem 4 (2 pts)**.** Let G be a graph on n vertices. Prove that $\delta(G) + \delta(\overline{G}) \leq n-1$ with equality if and only if G is regular.

Solution. Since $\deg_{\overline{G}} v = n - 1 - \deg_G v$, we observe that

$$\delta(\overline{G}) = \min_{v \in V(G)} (n - 1 - \deg_G(v)) = n - 1 - \max_{v \in V(G)} \deg_G(v) = n - 1 - \Delta(G).$$

Thus, since $\delta(G) \leq \Delta(G)$,

$$\delta(G) + \delta(\overline{G}) = \delta(G) + n - 1 - \Delta(G) \leq n - 1,$$

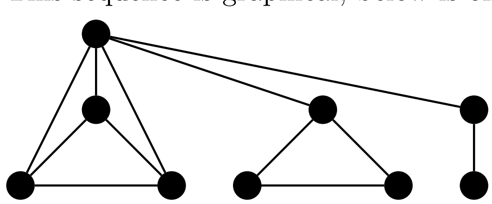
with equality if and only if $\delta(G) = \Delta(G)$, which happens precisely when G is regular. □

Problem 5 (1 + 1 pts)**.** Determine whether or not the following sequences are graphical. If the sequence is graphical, draw a picture of a graph with that degree sequences. If the sequence is not graphical, prove this.

1. $5, 3, 3, 3, 3, 2, 2, 2, 1$
2. $6, 5, 5, 4, 3, 2, 1$

Solution.

1. This sequence is graphical, below is one possible realization:



2. This sequence is not graphical. To see this, we repeatedly apply the Havel–Hakimi algorithm:

$$(6,5,5,4,3,2,1) \rightsquigarrow (4,4,3,2,1,0) \rightsquigarrow (3,2,1,0,0) \rightsquigarrow (1,0,0,-1).$$

The last sequence clearly isn't graphical, so Havel–Hakimi tells us that the original sequence is not graphical either.

Alternatively, we could apply Erdős–Gallai. Take $k = 3$ and so

$$\sum_{i=1}^{3} d_i = 6 + 5 + 5 = 16.$$

However,

$$k(k-1) + \sum_{i=4}^{7} \min\{3, d_i\} = 6 + 3 + 3 + 2 + 1 = 15 < 16,$$

and so the sequence cannot be graphical.

□