These solutions are from https://mathematicaster.org/teaching/graphs2022/sol-hw4.pdf

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (2 pts)**.** Let G and H be graphs. A *graph homomorphism* from G to H is a function $f\colon V(G) \to V(H)$ such that $\{f(u), f(v)\} \in E(H)$ whenever $\{u, v\} \in E(G)$.[1]

Prove that a graph G is bipartite if and only if there is a graph homomorphism from G to K_2.

Solution. Label $V(K_2) = \{v_1, v_2\}$.

($\Rightarrow$) Suppose that G is bipartite with parts A_1 and A_2 and consider the function $f\colon V(G) \to V(K_2)$ defined by

$$f(x) = \begin{cases} v_1 & \text{if } x \in A_1, \\ v_2 & \text{if } x \in A_2. \end{cases}$$

We claim that f is a graph homomorphism. Indeed, fix any edge $xy \in E(G)$; we may suppose that $x \in A_1$ and $y \in A_2$. Thus, $f(x) = v_1$ and $f(y) = v_2$ and so $f(x)f(y) \in E(K_2)$ as claimed.

($\Leftarrow$) Suppose that $f\colon V(G) \to V(K_2)$ is a graph homomorphism. Set $A_1 = f^{-1}(v_1)$ and $A_2 = f^{-1}(v_2)$; we claim that A_1, A_2 is a bipartition of G. To begin, certainly $V(G) = A_1 \sqcup A_2$, so we must show that A_1, A_2 are independent sets. Suppose for the sake of contradiction that there is an edge $uv \in E(G)$ with $u, v \in A_i$ (for $i \in [2]$). But then $f(u) = f(v) = v_i$ and so $v_iv_i \in E(K_2)$ since f is a graph homomorphism, which is impossible. □

Problem 2 (2 pts)**.** A graph G is called *self-complementary* if $G \cong \overline{G}$. For example, P_4 and C_5 are self-complementary.

Prove that if G is a self-complementary graph on n vertices, then n is congruent to either 0 or 1 modulo 4.

Solution. If $G \cong \overline{G}$, then certainly $|E(G)| = |E(\overline{G})|$; thus

$$|E(G)| = |E(\overline{G})| = \binom{n}{2} - |E(G)| \qquad \Longrightarrow \qquad |E(G)| = \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}.$$

Since $|E(G)|$ is an integer, it must be the case that $4 \mid n(n-1)$. Of course, one of n or $n-1$ is even and the other is odd, so the only way for this to happen is if $4 \mid n$ or if $4 \mid (n-1)$ The former means that $n \equiv 0 \pmod 4$ and the latter means that $n \equiv 1 \pmod 4$. □

Problem 3 (2 pts)**.** Prove that a graph G on n vertices with $\delta(G) \geq 3$ must contain a cycle of length at most $\lfloor n/2 \rfloor + 1$.

Solution. This proof is similar to that of HW2.4. Let $P = (v_1, \ldots, v_k)$ be a maximal path in G: since P is a path, we must have $k \leq n$. By the same logic used in HW2.4, since $\deg v_1 \geq 3$, we can find some $i < j \in \{3, \ldots, k\}$ such that $v_1v_i, v_1v_j \in E(G)$ or else we could extend P to a longer path (note that this implies that $k \geq 4$).

[1] Note that a graph homomorphism does *not* need to be a bijection and that it could be the case that $\{f(u), f(v)\} \in E(H)$ even though $\{u, v\} \notin E(G)$.

Therefore, both $C = (v_1, \dots, v_i)$ and $C' = (v_1, v_i, v_{i+1}, \dots, v_j)$ form cycles in G. Notice that the length of C is i and the length of C' is $j - i + 2$. If $i \leq \lfloor n/2 \rfloor + 1$, then C is our desired cycle, so suppose that $i \geq \lfloor n/2 \rfloor + 2$. But then C' has length

$$j - i + 2 \leq j + 2 - \left(\left\lfloor \frac{n}{2} \right\rfloor + 2 \right) = j - \left\lfloor \frac{n}{2} \right\rfloor \leq n - \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil \leq \left\lfloor \frac{n}{2} \right\rfloor + 1,$$

and so C' is our desired cycle. □

Problem 4 (2 pts)**.** Let T be a tree on n vertices with $\Delta(T) \leq 2$. Prove that $T \cong P_n$.

Solution. [#1] Suppose that $P = (v_1, \dots, v_k)$ is a maximum path in T. Set $V_k = \{v_1, \dots, v_k\}$ for ease of notation. We begin by claiming that $T[V_k] \cong P_k$. Indeed, the only way for this to fail is if there were some $i < j \in [k]$ with $j - i \geq 2$ with $v_i v_j \in E(T)$. But if this were the case, then $(v_i, v_{i+1}, \dots, v_j)$ would form a cycle in T, contradicting the fact that T is acyclic.

With this in hand, the claim will follow if we can show that $k = n$, so suppose for the sake of contradiction that $k < n$. In this case both V_k and $V(T) \setminus V_k$ are both nonempty sets and they partition $V(T)$, so, since T is connected, there must be an edge $xy \in E(T)$ with $x \in V_k$ and $y \in V(T) \setminus V_k$. We consider three cases (though two are basically the same case):

Case 1: $x \in \{v_2, \dots, v_{k-1}\}$. Suppose that $x = v_j$; then $v_{j-1}, v_{j+1}, y \in N(v_j)$ and so $\deg v_j \geq 3$ since these are three distinct vertices; this contradicts the fact that $\Delta(T) = 2$.

Case 2: $x = v_1$. Here we see that $(y, x = v_1, \dots, v_k)$ is a strictly longer path than P; contradiction.

Case 3: $x = v_k$. Here we see that $(v_1, \dots, v_k = x, y)$ is a strictly longer path than P; contradiction. □

Solution. [#2] We prove the claim by induction on n.

If $n = 1$, then the claim is obvious since the only trees on one vertex are isomorphic to P_1. (In fact, the same is true for $n \in \{2, 3\}$, but we don't need these to establish the base case).

Now suppose that $n \geq 2$. Since T is a tree on at least two vertices, it has a leaf, call it x. Also, let y be the unique neighbor of x in T. Now consider the tree $T' = T - x$. Certainly $\Delta(T') \leq \Delta(T) \leq 2$, and so the induction hypothesis implies that $T' \cong P_{n-1}$. We may therefore label $V(T') = \{v_1, \dots, v_{n-1}\}$ such that $E(T') = \{v_i v_{i+1} : i \in [n-2]\}$. Now, $y \in V(T')$ and so $y = v_i$ for some $i \in [n-1]$.

Suppose first that $i \in \{2, \dots, n-2\}$. In this case, we see that $v_{i-1}, v_{i+1}, x \in N_T(y)$ which means that $\deg_T y \geq 3$ since these are distinct vertices; a contradiction to the fact that $\Delta(T) \leq 2$. Therefore, $i \in \{1, n-1\}$. If $i = 1$, then T is the path $(x, v_1, \dots, v_{n-1})$ and if $i = n-1$, then T is the path $(v_1, \dots, v_{n-1}, x)$ since the only neighbor of x in T is $y = v_i$. In either case, we see that $T \cong P_n$ which concludes the proof. □

Problem 5 (2 pts)**.** Determine (with proof) all trees T (up to isomorphism) on $n \geq 2$ vertices such that $\overline{T}$ is also a tree. (Note: we do not require that $T \cong \overline{T}$.)

Solution. [#1] Since both T and $\overline{T}$ have n vertices and both are trees, they each have $n - 1$ edges. Thus,

$$n - 1 = |E(\overline{T})| = \binom{n}{2} - |E(T)| = \binom{n}{2} - (n-1) \implies 2(n-1) = \frac{n(n-1)}{2} \implies n = 4,$$

where the last implication follows since $n \geq 2$. Since $|V(T)| = 4$, we know that $Delta(T) \leq 3$, so we consider two cases:

Case 1: $\Delta(T) = 3$. Since T has 4 vertices, and 3 edges, this means that every edge is incident to that vertex of T with degree 3. This determines $T \cong K_{1,3}$. However, $\overline{K_{1,3}} \cong K_3 \sqcup K_1$, which is not a tree.

Case 2: $\Delta(T) \leq 2$. Problem 4 then implies that $T \cong P_4$, and we can check that P_4 is self-complementary and thus $\overline{T}$ is a tree as well!

We conclude that, up to isomorphism, the only tree T on at least two vertices with the property that $\overline{T}$ is also a tree is P_4. □

Solution. [#2] To begin, we notice that if $A \subseteq V(T)$ is an independent set of T, then A induces a clique in $\overline{T}$. Since cliques of size at least three have cycles, this implies that $|A| \leq 2$ since $\overline{T}$ is a tree. From here, we can see that $\Delta(T) \leq 2$. Indeed, for any $v \in V(T)$, we know that $N(v)$ is an independent set (or else T has a 3-cycle), and so $\deg v = |N(v)| \leq 2$. Problem 4 then implies that $T \cong P_n$ for some $n \geq 2$. We handle the various cases in turn.

Case 1: $n = 2$. Here, we see that $\overline{T} \cong \overline{K_2}$ which is not a tree.

Case 2: $n = 3$. Here, we see that $\overline{T} \cong K_1 \sqcup K_2$ which is not a tree.

Case 3: $n = 4$. $\overline{P}_4 \cong P_4$ so this works!

Case 4: $n \geq 5$. Label $V(T) = \{v_1, \ldots, v_n\}$ so that $E(T) = \{v_i v_{i+1} : i \in [n-1]\}$, which we can do since $T \cong P_n$. Now observe that $\{v_1, v_3, v_5\}$ is an independent set of T of size three, which contradicts our earlier observation.

We conclude that, up to isomorphism, the only tree T on at least two vertices with the property that $\overline{T}$ is also a tree is P_4. □