These solutions are from https://mathematicaster.org/teaching/graphs2022/sol-hw5.pdf

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (0.5 + 0.5 + 1 pts)**.** For a fixed integer $k \geq 3$, a graph G is said to have property $\mathcal{C}_k$ if every subgraph of G with at least k edges contains a cycle. For a fixed integer $k \geq 4$, a graph G is said to have property $\mathcal{E}_k$ if every subgraph of G with at least k edges contains an even cycle.

1. For each $k \geq 3$, construct a graph G which has property $\mathcal{C}_k$ and has $|E(G)| = \binom{k}{2}$.
2. For each $k \geq 4$, construct a graph G which has property $\mathcal{E}_k$ and has $|E(G)| = \lceil k/2 \rceil \lfloor k/2 \rfloor$.
3. Prove that if G is a connected graph with property $\mathcal{C}_k$, then $|E(G)| \leq \binom{k}{2}$.

Solution.

1. Consider $G = K_k$, which has $|E(G)| = \binom{k}{2}$ and $|V(G)| = k$. If H is any subgraph of G with at least k edges, then $|E(H)| \geq k = |V(G)| \geq |V(H)|$; thus H must contain a cycle.
2. Consider $G = K_{\lceil k/2 \rceil, \lfloor k/2 \rfloor}$ which has $|E(G)| = \lceil k/2 \rceil \lfloor k/2 \rfloor$ and $|V(G)| = k$. If H is any subgraph of G with at least k edges, then $|E(H)| \geq k = |V(G)| \geq |V(H)|$; thus H must contain a cycle. Now, H is a subgraph of G, which is bipartite, and so H is also bipartite. Therefore, this cycle contained in H must be of even length.
3. Suppose that G has n vertices. Then since G is connected, it contains a spanning tree T, which has $n-1$ edges. Since T has no cycles and G has property $\mathcal{C}_k$, we must have $n-1 < k$, and so $n \leq k$. Therefore, $|E(G)| \leq \binom{n}{2} \leq \binom{k}{2}$ as desired.

N.b. Part 3 remains true even if G is disconnected (this is not too difficult to show, but it does require a sneaky idea). Additionally, Part 2 is tight; that is, if G is any graph with property $\mathcal{E}_k$, then $|E(G)| \leq \lceil k/2 \rceil \lfloor k/2 \rfloor$ (this isn't an easy result). If you're curious about either of these, take a look at Theorem 3.8, Theorem 1.2 and Lemma 3.10 in the paper https://arxiv.org/abs/1711.02082. □

Problem 2 (3 pts)**.** This problem expands on the observation that trees on at least two vertices have at least two leaves.

Let T be a tree on at least two vertices. Let $\ell(T)$ denote the number of leaves of T and define $D_{\geq 2} = \{v \in V(T) : \deg v \geq 2\}$. Prove that

$$\ell(T) = 2 + \sum_{v \in D_{\geq 2}} (\deg v - 2)$$

Solution. [#1] Let L denote the set of leaves of T; note that since T has at least two vertices (and hence has no isolated vertices), we have $V(T) = L \sqcup D_{\geq 2}$. Now,

$$\ell(T) = |L| = \sum_{v \in L} 1 = \sum_{v \in L} (2 - \deg v) = -\sum_{v \in L} (\deg v - 2).$$

We next invoke the handshaking lemma to find that

$$\sum_{v \in V(T)} (\deg v - 2) = \sum_{v \in V(T)} \deg v - \sum_{v \in V(T)} 2 = \sum_{v \in V(T)} \deg v - 2|V(T)| = 2|E(T)| - 2|V(T)| = -2.$$

Therefore,

$$-2 = \sum_{v \in V(T)} (\deg v - 2) = \sum_{v \in D_{\geq 2}} (\deg v - 2) + \sum_{v \in L} (\deg v - 2) = \sum_{v \in D_{\geq 2}} (\deg v - 2) - \ell(T),$$

from which the claim follows. $\square$

Solution. [#2] We prove the claim by induction on $n = |V(T)|$.

The base case is $n = 2$, in which case we must have $T \cong K_2$. Thus $\ell(T) = 2$ and $D_{\geq 2} = \varnothing$ and so the claim holds.

Now suppose that $n \geq 3$. Fix any leave $x \in V(T)$ (we know x exists since T has at least two leaves) and let $y \in V(T)$ be the unique neighbor of x. Now, consider the tree $T' = T - x$ and define $D'_{\geq 2} = \{v \in V(T') : \deg_{T'} v \geq 2\}$. Since T' has $n - 1 \geq 2$ vertices, we can apply the induction hypothesis to T' to find that

$$\ell(T') = 2 + \sum_{v \in D'_{\geq 2}} (\deg_{T'} v - 2).$$

Observe that $\deg_{T'} v = \deg_T v$ for all $v \in V(T') \setminus \{y\}$ and that $\deg_{T'} y = \deg_T y - 1$. In particular, by ignoring the vertex y (and noting that certainly $x \notin D_{\geq 2}$, we have

$$\sum_{v \in D'_{\geq 2} \setminus \{y\}} (\deg_{T'} v - 2) = \sum_{v \in D_{\geq 2} \setminus \{y\}} (\deg_T v - 2).$$

We now consider adding back the vertex x; here we must consider what happens with the vertex y.

Case 1: $\deg_{T'} y \geq 2$. Here, we certainly have $y \in D'_{\geq 2}$ and $y \in D_{\geq 2}$ and so

$$\sum_{v \in D'_{\geq 2}} (\deg v - 2) = \sum_{v \in D_{\geq 2}} (\deg v - 2) + 1;$$

thus we must show that $\ell(T) = \ell(T') + 1$. But this is clear: x is a leaf of T and y is *not* a leaf of T'.

Case 1: $\deg_{T'} y = 1$. Here, we have $y \notin D'_{\geq 2}$ yet $y \in D_{\geq 2}$. However, $\deg_T y - 2 = 0$ and so

$$\sum_{v \in D'_{\geq 2}} (\deg v - 2) = \sum_{v \in D_{\geq 2}} (\deg v - 2);$$

thus we must show that $\ell(T) = \ell(T')$. To see this, we know that y is a leaf of T', but is not a leaf of T; however, x is a leaf of T. $\square$

Problem 3 (3 pts)**.** Fix an integer $n \geq 2$. Prove that a sequence of integers $d_1, \ldots, d_n$ is the degree sequence of some tree if and only if $d_i \geq 1$ for all $i \in [n]$ and $\sum_{i=1}^{n} d_i = 2n - 2$.

Solution. [#1] ($\Rightarrow$) Suppose that $d_1, \dots, d_n$ is the degree sequence of a tree T. Since $n \geq 2$, we know that $\delta(T) \geq 1$ (or else T is disconnected) and so $d_i \geq 1$ for all $i \in [n]$. Then the handshaking lemma implies that

$$\sum_{i=1}^{n} d_i = 2|E(T)| = 2(n-1) = 2n-2.$$

($\Leftarrow$) We prove the claim by induction on $n \geq 2$.

For the base case of $n = 2$, we must have $d_1 + d_2 = 2$ if and only if $d_1 = d_2 = 1$ since $d_1, d_2 \geq 1$. Note that this is exactly the degree sequence of K_2 which is a tree.

Now suppose that $n \geq 3$. Without loss of generality, we may suppose that $d_1 \geq \cdots \geq d_n$. Therefore,

$$nd_n \leq \sum_{i=1}^{n} d_i \leq nd_1.$$

In particular, $nd_n \leq 2n - 2$ and so $d_n \leq 2 - (2/n) < 2$. This implies that $d_n = 1$ since $d_n \geq 1$ by assumption. Similarly, $nd_1 \geq 2n - 2$ and so $d_1 \geq 2 - (2/n) > 1$ since $n \geq 3$ and so $d_1 \geq 2$.

Now, consider the sequence $\gamma_1, \dots, \gamma_{n-1}$ where $\gamma_1 = d_1 - 1$ and $\gamma_i = d_i$ for all $i \in [n-1]$. From above, we know that $\gamma_i \geq 1$ for all $i \in [n-1]$. Additionally, since $d_n = 1$, we have

$$\sum_{i=1}^{n-1} \gamma_i = (d_1 - 1) + \sum_{i=2}^{n-1} d_i = \left(\sum_{i=1}^{n} d_i\right) - 2 = 2n - 4 = 2(n-1) - 2.$$

Thus, by the induction hypothesis, we can find a tree T with vertex set $V(T) = \{v_1, \dots, v_{n-1}\}$ such that $\deg v_i = \gamma_i$. From here, we build a new tree T' by appending a new vertex v_n which is adjacent to only v_1. Certainly T' is a tree since T is a tree and we only added a new leaf. Furthermore, $\deg_{T'} v_n = 1 = d_n$, $\deg_{T'} v_1 = \deg_T v_1 = \gamma_1 + 1 = d_1$ and $\deg_{T'} v_i = \gamma_i = d_i$ for all $i \in \{2, \dots, n-1\}$. Therefore, T' has degree sequence $d_1, \dots, d_n$ and we have established the claim. □

Solution. [#2] ($\Rightarrow$) Same as in the first solution.

($\Leftarrow$) Without loss of generality, we may suppose that $d_1 \geq \cdots \geq d_n$. Fix $V = \{v_1, \dots, v_n\}$ for notational ease. Roughly speaking, we show that we can build the desired tree greedily, vertex by vertex.

In order to show this is possible, we prove the following claim. (It is best to understand the proof of the claim as an algorithm for building the tree)

Claim 1. *For each $k \in [n]$, there is a graph F on vertex set V with the following properties:*

1. *$F[\{v_1, \dots, v_k\}]$ is a tree.*
2. *$\deg_F v_i \leq d_i$ for all $i \in [n]$, and*
3. *$\deg_F v_i = 0$ for all $i \in \{k+1, \dots, n\}$, and*
4. *If $k = n$, then $\deg_F v_i = d_i$ for all $i \in [n]$. If $k < n - 1$, then there is some $i \in [k]$ for which $\deg_F v_i \leq d_i - 1$.*

Proof. We prove the claim by induction on k.

For the base case of $k = 1$, we can take $F = (V, \varnothing)$ which clearly satisfies all properties since $d_1 \geq 1$.

Now suppose that $k \in \{2, \ldots, n-1\}$. By the induction hypothesis, there is a graph F_{k-1} which satisfies the conditions for the value $k-1 \geq 1$. Let $j \in [k-1]$ be such that $\deg_{F_{k-1}} v_j \leq d_j - 1$, which is guaranteed to exist. We build a graph F by adding the edge $v_k v_j$ to F_{k-1}. We claim that F is the graph we're looking for.

1. Observe that $F[\{v_1, \ldots, v_{k-1}\}] = F_{k-1}[\{v_1, \ldots, v_{k-1}\}]$ since we only added the edge $v_j v_k$ for some $j \in [k-1]$. Thus, $F[\{v_1, \ldots, v_{k-1}\}]$ is a tree by the induction hypothesis. Then $F[\{v_1, \ldots, v_k\}]$ is formed by adding a leaf to $F[\{v_1, \ldots, v_{k-1}\}]$ and so it is a tree as desired.

2. Notice that $\deg_F v_k = 1 \leq d_k$ and that $\deg_F v_j = \deg_{F_{k-1}} v_j + 1 \leq d_j$. Furthermore, for all $i \in [n] \setminus \{j, k\}$, we have $\deg_F v_i = \deg_{F_{k-1}} v_i \leq d_i$ by the induction hypothesis.

3. For $i \in \{k+1, \ldots, n\}$, we have $\deg_F v_i = \deg_{F_{k-1}} v_i = 0$ by the induction hypothesis.

4. This is the most involved step.

   Assume first that $k = n$, in which case we would have $F[\{v_1, \ldots, v_k\}] = F[V] = F$. Property 1 then says that F is a tree and so, by using property 2, we have

   $$2(n-1) = \sum_{i=1}^{n} \deg_F v_i \leq \sum_{i=1}^{n} d_i = 2n - 2.$$

   Thus we must have equality throughout and so $\deg_F v_i = d_i$ for all $i \in [n]$ as needed.

   Now, we know that $\deg_F v_i \leq d_i$ for all $i \in [k]$ (property 2), the only way that there is *no* $i \in [k]$ for which $\deg_F v_i \leq d_i - 1$ is if $\deg_F v_i = d_i$ for all $i \in [k]$. We show that if this is the case, then $k = n$, which will satisfy the claim.

   Since $\deg_F v_i = d_i$ for all $i \in [k]$, we know that $d_k = \deg_F v_k = 1$. In particular, $d_i = 1$ for all $i \in \{k, \ldots, n\}$ since we have $d_1 \geq \cdots \geq d_n \geq 1$.

   Now, $F[\{v_1, \ldots, v_k\}]$ is a tree and so it has exactly $k-1$ edges. Additionally, $\deg_F v_i = 0$ for all $i \in \{k+1, \ldots, n\}$ and so all edges of F are among $\{v_1, \ldots, v_k\}$. We thus apply the handshaking lemma to conclude that

   $$2(k-1) = \sum_{i=1}^{k} \deg_F v_i = \sum_{i=1}^{k} d_i.$$

   Recalling that $d_i = 1$ for all $i \in \{k+1, \ldots, n\}$ we see that

   $$2n - 2 = \sum_{i=1}^{n} d_i = (n-k) + 2(k-1) \implies 2n = n + k \implies k = n,$$

   as needed. □

To finish the proof of the problem, let F be the graph guaranteed by the $k = n$ case of Claim 1. Then $F[\{v_1, \ldots, v_n\}] = F$ is a tree by the first property and $\deg_F v_i = d_i$ for all $i \in [n]$ by the last property. Thus F is a tree with degree sequence $d_1, \ldots, d_n$, just like we wanted. □

Solution. [#3] (⇒) Same as in the first solution.

(⇐) We prove the following two claims, which are more general than strictly necessary for this problem.

Claim 2. *If $d_1 \geq \cdots \geq d_n \geq 1$ and $\sum_{i=1}^n d_i = 2k$ for some $k \in [n-1]$, then $d_1, \ldots, d_n$ is graphical.*

Proof #1. We employ the Erdős–Gallai conditions. Certainly $\sum_{i=1}^n d_i = 2k$ is even, so fix any $r \in [n]$; we must show that $\sum_{i=1}^r d_i \leq r(r-1) + \sum_{i=r+1}^n \min\{r, d_i\}$. We do so via the following train of inequalities:

$$\begin{aligned}\sum_{i=1}^r d_i &= 2k - \sum_{i=r+1}^n d_i \leq 2k - (n-r) \leq 2(n-1) - (n-r) = n + r - 2 \\ &= 2(r-1) + (n-r) \leq 2(r-1) + \sum_{i=r+1}^n \min\{r, d_i\} \\ &\leq r(r-1) + \sum_{i=r+1}^n \min\{r, d_i\}.\end{aligned}$$

□

Proof #2. We prove this by induction on k. For a base-case, consider $k = 1$ and any $n \geq 2$. Then $\sum_{i=1}^n d_i = 2$ and $d_1 \geq \cdots \geq d_n \geq 1$ imply that $n = 2$ and $d_1 = d_2 = 1$, which is the degree sequence of K_2.

Now consider $k \geq 2$. We first notice that $2k = \sum_{i=1}^n d_i \geq d_1 + (n-1)$ and so $d_1 \leq 2k - (n-1) \leq n-1$ since $k \in [n-1]$. Thus, consider the sequence $\gamma_2, \ldots, \gamma_n$ where $\gamma_i = d_i - 1$ if $i \leq d_1 + 1$ and $\gamma_i = d_i$ if $i > d_1 + 1$ (i.e. the sequence used in the Havel–Hakimi algorithm). We claim that $\gamma_2, \ldots, \gamma_n$ is graphical. Suppose that there are z zero values among $\gamma_2, \ldots, \gamma_n$; note that $z \leq d_1$ since $d_i \geq 1$ for all i. If $z = n-1$, then $\gamma_2, \ldots, \gamma_n$ is the 0 sequence, which is graphical; thus suppose that $z < n-1$. Observe that

$$\sum_{i=2}^n \gamma_i = \sum_{i=2}^{d_1+1} (d_i - 1) + \sum_{i=d_1+2}^n d_i = \left(\sum_{i=2}^n d_i\right) - d_1 = \left(\sum_{i=1}^n d_i\right) - 2d_1 = 2(k - d_1).$$

We claim that $1 \leq k - d_1 \leq (n-1-z) - 1$. Firstly, we know that $d_1 \leq 2k - (n-1) \leq k$ since $k \leq n-1$ and so $k - d_1 \geq 0$. Also, if $k - d_1 = 0$, then $\gamma_2, \ldots, \gamma_n$ is the zero sequence, so $k - d_1 \geq 1$. Now, $z \leq d_1$ and so $n - 1 - z \geq n - 1 - d_1 \geq k - d_1$. If $k - d_1 = n - 1 - z$, then we must have $k = n-1$ and $d_1 = z$. Note that then $\gamma_2 = \cdots = \gamma_{d_1+1} = 0$ and so $d_2 = \cdots = d_{d_1+1} = 1$, which implies that $d_2 = \cdots = d_n = 1$. Thus, $2(n-1) = \sum_{i=1}^n d_i = d_1 + (n-1) = z + (n-1) \implies z = n-1$ and so $\gamma_2, \ldots, \gamma_n$ is the zero sequence again.

Putting these observations together, we have $k - d_1 \in [(n-z-1)-1]$ and certainly $k - d_1 < k$ since $d_1 \geq 1$. Thus, we may apply the induction hypothesis to the $n - z - 1$ many γ_i's which are non-zero to find a realization of that sequence. Adding z isolated vertices to this realization is then a realization of $\gamma_2, \ldots, \gamma_n$. In any case, $\gamma_2, \ldots, \gamma_n$ is graphical and so $d_1, \ldots, d_n$ is also graphical thanks to Havel–Hakimi. □

Claim 3. *If $d_1 \geq \cdots \geq d_n \geq 1$ is any graphical sequence with $\sum_{i=1}^n d_i \geq 2(n-c)$ where c is some positive integer, then there is a realization which has at most c connected components.*

Proof. Since $d_1, \ldots, d_n$ is assumed to be graphical, there is at least one realization. Let G be a realization with the fewest number of connected components. Suppose that the connected components of G are $G_1, \ldots, G_k$; we wish to show that $k \leq c$, so suppose for the sake of contradiction that $k \geq c+1$. Since $\sum_{i=1}^n d_i \geq 2(n-c)$, the handshaking lemma tells us that G has at least $n-c$ many edges. If G were acyclic, then $|E(G)| = n - k \leq n - (c+1) < n - c$, so we know that G contains a cycle. Without loss of generality, G_1 contains a cycle; let x and y be any pair of adjacent vertices on this cycle. Now consider G_2; since $d_i \geq 1$ for each i, we know that G_2 contains an edge, call one of them uv. Thus, xy and uv are edges of G and xu and yv are non-edges of G. Consider the 2-switch where we replace xy and uv by xu and yv and call the resulting graph G'. Since xy was chosen to be an edge of a cycle, we observe that G' has $k-1$ connected components (the 2-switch "merged" G_1 and G_2); a contradiction since G' is also a realization of $d_1, \ldots, d_n$. □

Now to finish the actual problem: without loss of generality, we may suppose that $d_1 \geq \cdots \geq d_n \geq 1$. By assumption $\sum_{i=1}^{n} d_i = 2(n-1)$ and so Claim 2 implies that $d_1, \ldots, d_n$ is graphical. Then Claim 3 implies that there is a realization G with at most one connected component (i.e. G is connected). Finally, the handshaking lemma implies that G has exactly $n-1$ edges and so G is a tree.

In fact, we have shown more: $d_1, \ldots, d_n$ is the degree sequence of a forest with no isolated vertices if and only if $d_i \geq 1$ for all $i \in [n]$, and the sum $\sum_{i=1}^{n} d_i$ is even and bounded above by $2n-2$. □

Problem 4 (2 pts)**.** Let G be a connected graph and let $w\colon E(G) \to \mathbb{R}$ be a weight function. Show that if all weights are distinct (that is $w(e) \neq w(s)$ for all distinct $e, s \in E(G)$), then G has a *unique* minimum spanning tree.

Solution. We know that G has at least one minimum spanning tree since G is connected. Thus suppose for the sake of contradiction that there are at least two distinct minimum spanning trees, call two of them T_1 and T_2. Since T_1 and T_2 are different, we know that $E(T_1) \neq E(T_2)$. In particular, the symmetric difference $E(T_1) \triangle E(T_2)$ is nonempty.[1] Let $e \in E(T_1) \triangle E(T_2)$ be the smallest weight edge in this set. Without loss of generality, we may suppose that $e \in E(T_1) \setminus E(T_2)$. Now, consider the graph $H = T_2 + e$; since $e \notin E(T_2)$ and T_2 is a tree, H must contain a cycle C which uses the edge e. Of course, T_1 is a tree and so there must be some edge $s \in E(C) \setminus E(T_1)$; note that $s \in E(T_2)$. Thus set $T_3 = H - s = T_2 + e - s$, which is a spanning tree since we removed an edge from the cycle C in H.

Now, $s \in E(T_2) \setminus E(T_1) \subseteq E(T_1) \triangle E(T_2)$ and so $w(s) > w(e)$ since e was chosen to be the smallest weight element of this set and all weights are distinct. But then T_3 is a spanning tree with $w(T_3) = w(T_2) + w(e) - w(s) < w(T_2)$ which contradicts the fact that T_2 is a minimum weight spanning tree. □

[1] Recall that $X \triangle Y = (X \setminus Y) \cup (Y \setminus X) = (X \cup Y) \setminus (X \cap Y)$.