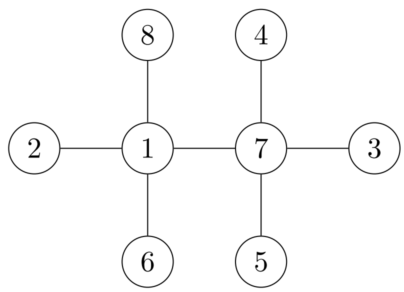
These solutions are from https://mathematicaster.org/teaching/graphs2022/sol-hw6.pdf

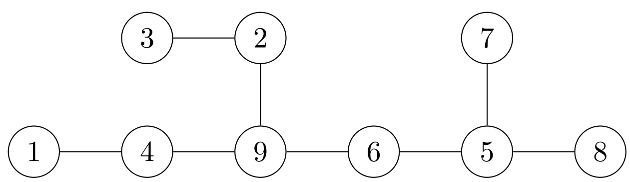
Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (0.5 + 0.5 pts)**.** Determine the Prüfer code of the following trees (using the standard ordering of the integers).

1.



2.



Solution.

1. $(1, 7, 7, 7, 1, 1)$
2. $(4, 2, 9, 9, 5, 5, 6)$

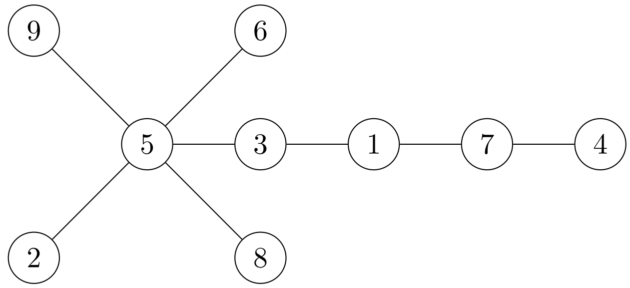
□

Problem 2 (0.5 + 0.5 pts)**.** For the following sequences, draw a picture of the (labeled) tree which has that sequence as its Prüfer code (using the standard ordering of the integers). Your tree should have vertex-set $[n]$ for some integer n.

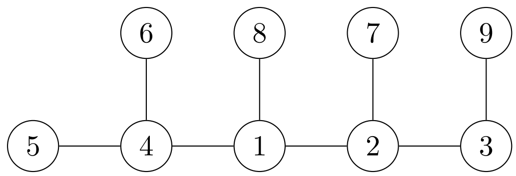
1. $(5, 7, 5, 1, 3, 5, 5)$
2. $(4, 4, 1, 2, 1, 2, 3)$

Solution.

1.



2.



□

Problem 3 (1 pts)**.** Determine (with proof) all trees T (up to isomorphism) on $n \geq 2$ vertices whose Prüfer code uses each element of $V(T)$ at most once (under any arbitrary ordering of $V(T)$).

Solution. Under any ordering of $V(T)$, the symbol $x \in V(T)$ appears exactly $\deg x - 1$ many times in the Prüfer code of T (Lemma 2 from the notes). Therefore, x appears at most once in the Prüfer code of T if and only if $\deg x \leq 2$. Phrased differently, the Prüfer code of T sees each element of $V(T)$ at most once if and only if $\Delta(T) \leq 2$. This happens if and only if $T \cong P_n$ as per HW3.4. □

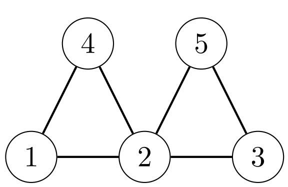
Problem 4 (2 pts)**.** (HW5.3 revisited) Fix an integer $n \geq 2$ and let $d_1, \ldots, d_n$ be a sequence of positive integers with $\sum_{i=1}^{n} d_i = 2n - 2$. Use Prüfer codes to show that there is a tree with degree sequence $d_1, \ldots, d_n$.

Solution. Since $\sum_{i=1}^{n} d_i = 2n - 2$, we have that $\sum_{i=1}^{n}(d_i - 1) = n - 2$. Additionally, $d_i - 1$ is a non-negative integer since d_i is a positive integer. We can therefore find a sequence of integers $P \in [n]^{n-2}$ such that, for each $i \in [n]$, the element i appears exactly $d_i - 1$ times in P. Indeed, simply write 1 down $d_1 - 1$ many times, 2 down $d_2 - 1$ many times, etc.

Let T be the tree on vertex-set $[n]$ whose Prüfer code is P. Then Lemma 2 from the notes implies that $\deg_T i = d_i$ for each $i \in [n]$; hence T has degree sequence $d_1, \ldots, d_n$, as needed. □

Problem 5 (2 pts)**.** For a graph G, define the relation R on $V(G)$ by $u \, R \, v$ if and only if $u = v$ or there is a cycle in G containing both u and v. Find a graph G wherein R is *not* an equivalence relation on $V(G)$. (You are welcome to define G via a picture, though, of course, you must still demonstrate that R is not an equivalence relation on this G)

Solution. Consider the following graph:



Observe that $1\,R\,2$ as witnessed by the cycle $(1,2,4)$, and $2\,R\,3$ as witnessed by the cycle $(2,3,5)$. However, there is no cycle containing both 1 and 3 and so $(1,3) \notin R$. Thus R is not an equivalence relation on this particular graph since R is not transitive. □

Problem 6 (3 pts)**.** For a graph G, define the relation R on $E(G)$ by $e\,R\,s$ if and only if $e = s$ or there is a cycle in G containing both e and s. Prove that R is an equivalence relation on $E(G)$.

Solution. Reflexivity is obvious since $e = e$ for any edge $e \in E(G)$. Symmetry is also obvious since if there is a cycle containing both e and s, then this same cycle contains both s and e.

Transitivity is the only interesting part. Suppose that $e\,R\,s$ and $s\,R\,t$; we must show that $e\,R\,t$. If either $e = s$ or $s = t$, then this is immediate, so we may suppose that e, s, t are three distinct edges. Thus, let C_{es} be a cycle in G containing both e and s and let C_{st} be a cycle in G containing both s and t. Label $C_{es} = (v_1, \dots, v_k)$ where $e = v_1 v_k$ and label $C_{st} = (u_1, \dots, u_\ell)$ where $t = u_1 u_\ell$; of course, $k, \ell \geq 3$. Let i be the smallest index for which $v_i \in \{u_1, \dots, u_\ell\}$ and let j be the largest index for which $v_j \in \{u_1, \dots, u_\ell\}$; we begin by showing that both i, j exist and that $i < j$. Indeed, s belongs to both C_{es} and to C_{st} and so $s = v_a v_{a+1}$ and $s = u_b u_{b+1}$ for some $a \in [k-1]$ and some $b \in [\ell - 1]$ (since $e \neq s$ and $t \neq s$). In particular, $v_a, v_{a+1} \in \{u_1, \dots, u_\ell\}$ and so $1 \leq i \leq a < a+1 \leq j \leq k$. Suppose that $v_i = u_{i'}$ and $v_j = u_{j'}$ for some $i', j' \in [\ell]$; of course $i' \neq j'$. By the definition of i and j, we know that $\{v_1, \dots, v_{i-1}\} \cap \{u_1, \dots, u_\ell\} = \{v_k, \dots, v_{j+1}\} \cap \{u_1, \dots, u_\ell\} = \varnothing$.

If $i' < j'$, consider the sequence $C = (v_1, \dots, v_i = u_{i'}, u_{i'-1}, \dots, u_1, u_\ell, u_{\ell-1}, \dots, u_{j'} = v_j, v_{j+1}, \dots, v_k)$ and if $i' > j'$, consider the sequence $C = (v_1, \dots, v_i = u_{i'}, u_{i'+1}, \dots, u_\ell, u_1, u_2, \dots, u_{j'} = v_j, v_{j+1}, \dots, v_k)$. By the earlier remark, in either case, C forms a cycle. Furthermore, since $e = v_1 v_k$ and $t = u_1 u_\ell$, C forms a cycle in G containing both e and t; thus $e\,R\,t$ as needed. □