These solutions are from https://mathematicaster.org/teaching/graphs2022/sol-hw7.pdf

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (2pts)**.** For infinitely many integers n, construct a graph G with the following properties:

- G is connected and has n vertices, and
- G is *not* a tree, and
- Every $v \in V(G)$ which is not a leaf of G is a cut-vertex of G.

"For infinitely many integers n..." means the following: Pick your favorite infinite subset $A \subseteq \mathbb{N}$ and, for each $n \in A$, build the desired graph for that n. For instance, maybe you pick A to be the set of even naturals, or maybe you pick A to be the set of integers of the form 2^k, or maybe you pick A to be the set of all primes larger than $100^{100^{100^{100}}}$, etc. As long as A is infinite, you're fine.

Solution. There are tons of examples; here's just one such example which is easy to describe. Fix any even integer $n \geq 6$ and set $m = n/2$ so $m \geq 3$ is an integer. Consider building the graph G by attaching a unique leaf to each vertex of C_m. In particular, G has vertex-set $\{v_1, \ldots, v_m, u_1, \ldots, u_m\}$ where $G[\{v_1, \ldots, v_m\}]$ is a copy of C_m and, for each $i \in [m]$, the unique neighbor of u_i is v_i. Now, G has $2m = n$ many vertices and is connected by construction. Also, G is not a tree since it contains an m-cycle. Now, observe that each u_i is a leaf of G since the only neighbor of u_i is v_i. On the other hand, $\deg v_i = 3$ for all i; that is, the non-leaves of G are $v_1, \ldots, v_m$. Now, for any $i \in [m]$, observe that u_i is an isolated vertex in $G - v_i$; thus $G - v_i$ is not connected. Hence G has the property that every non-leaf is a cut-vertex as desired. □

Problem 2 (1 + 1 pts)**.** Let G be a graph and suppose that H is any spanning subgraph of G.

1. Prove that $\lambda(G) \geq \lambda(H)$.
2. Prove that $\kappa(G) \geq \kappa(H)$.

Solution. [#1]

1. If $G \cong K_1$, then also $H \cong K_1$. Thus, $\lambda(G) = \lambda(H) = 0$.

   Otherwise, let $S \subseteq E(G)$ be a minimum edge-cut of G. Since $G - S$ is disconnected, we can partition $V(G) = A \sqcup B$ with A, B non-empty so that no edge of $G - S$ crosses between A and B. Now, $V(G) = V(H)$ since H is a spanning subgraph of G, and so A, B also form a partition of $V(H)$ into non-empty pieces. Set $S' = S \cap E(H)$. Since $E(H - S') \subseteq E(G - S)$, we know that no edge of $H - S'$ crosses between A and B. Thus S' is an edge-cut of H and so $\lambda(H) \leq |S'| \leq |S| = \lambda(G)$ as desired.

2. If $G \cong K_n$, then $\kappa(G) = n - 1$. Also, H has $V(H) = V(G)$ and so $\kappa(H) \leq n - 1$ as well. Thus, going forward, we may suppose that G is not a clique.

Thus, let $U \subseteq V(G)$ be a minimum vertex-cut of G. Since $G - U$ is disconnected, we can partition $V(G) \setminus U = A \sqcup B$ with A, B non-empty so that no edge of $G - U$ crosses between A and B. Now, $V(G) = V(H)$ since H is a spanning subgraph of G and so A, B also form a partition of $V(H) - U$ into non-empty pieces. Thus, U is also a vertex-cut of H and so $\kappa(H) \leq |U| = \kappa(G)$.

□

Solution. [#2]

1. Fix any $u \neq v \in V(H) = V(G)$ (spanning subgraph). Menger's theorem for edge-connectivity tells us that H contains at least $\lambda(H)$ many edge-disjoint u-v paths. Since H is a subgraph of G, these paths also exist in G and so G contains at least $\lambda(H)$ many edge-disjoint u-v paths as well. Thus, since $V(H) = V(G)$, Menger's theorem for edge-connectivity tells us that $\lambda(G) \geq \lambda(H)$.

2. Fix any $u \neq v \in V(H) = V(G)$ (spanning subgraph). Menger's theorem for vertex-connectivity tells us that H contains at least $\kappa(H)$ many internally-disjoint u-v paths. Since H is a subgraph of G, these paths also exist in G and so G contains at least $\kappa(H)$ many edge-disjoint u-v paths as well. Thus, since $V(H) = V(G)$, Menger's theorem for vertex-connectivity tells us that $\kappa(G) \geq \kappa(H)$.

□

Problem 3 (1 + 2 pts)**.** Let $G = (V, E)$ be a graph with at least one edge and fix any $e \in E$.

1. Prove that $\lambda(G) \geq \lambda(G - e) \geq \lambda(G) - 1$.

2. Prove that $\kappa(G) \geq \kappa(G - e) \geq \kappa(G) - 1$.

Solution. [#1]

1. Since $G - e$ is a spanning subgraph of G, Problem 2 implies that $\lambda(G) \geq \lambda(G - e)$.

   In this case, we don't need to worry about K_1 since it doesn't have any edges. So let S be a minimum edge-cut of $G - e$, so $|S| = \lambda(G - e)$. Then $S \cup \{e\}$ is an edge-cut of G and so $\lambda(G) \leq |S| + 1 = \lambda(G - e) + 1 \implies \lambda(G - e) \geq \lambda(G) - 1$.

2. We consider first the case when $G \cong K_n$ for some $n \geq 2$. Here we have $\kappa(G) = n - 1$ and $\kappa(G - e) = n - 2$, which satisfies the claim. Thus, going forward, we can assume that G is not a clique.

   Now, since $G - e$ is a spanning subgraph of G, Problem 2 implies that $\kappa(G) \geq \kappa(G - e)$.

   For the reverse inequality, since G is not a clique, certainly $G - e$ is also not a clique. Thus let U be a minimum vertex-cut of $G - e$, so $|U| = \kappa(G - e)$. Since U is a vertex-cut of $G - e$, this means that we can partition $V((G - e) - U) = A \sqcup B$ with A, B non-empty such that no edge of $(G - e) - U$ crosses between A and B.

   If $G - U$ is disconnected, then U is a vertex-cut of G and so $\kappa(G - e) = |U| \geq \kappa(G) \geq \kappa(G) - 1$. Thus, suppose that $G - U$ is connected. Since $V(G - e) = V(G)$, the sets A, B are also a partition of $V(G - U)$ into non-empty parts, so since $G - U$ is connected, then there is some edge of $G - U$ which crosses between A and B. The only option here is the edge e. Now, without loss of generality, we may suppose that $|A| \geq |B|$. If we were to have $|A| = 1$, then we would have $\kappa(G - e) = |U| = n - 2$. Since $\kappa(G) \leq n - 1$ always, certainly we would have $\kappa(G - e) \geq \kappa(G) - 1$. Thus, we may suppose that $|A| \geq 2$. Let a be any vertex of A which is an end-point of e, which is possible since $|A| \geq 2$ and e has exactly one end-point in A. Then $G - (U \cup \{a\})$ is disconnected since $V(G - (U \cup \{a\})) = (A \setminus \{a\}) \sqcup B$, both $A \setminus \{a\}, B$ are non-empty and there is no edge between A and B. Thus, $\kappa(G) \leq |U \cup \{a\}| = |U| + 1 = \kappa(G - e) + 1 \implies \kappa(G - e) \geq \kappa(G) - 1$.

□

Solution. [#2]

1. Since $G - e$ is a spanning subgraph of G, Problem 2 implies that $\lambda(G) \geq \lambda(G - e)$, thus we need only prove that $\lambda(G - e) \geq \lambda(G) - 1$.

   Fix any $u \neq v \in V(G) = V(G - e)$. Menger's theorem for edge-connectivity tells us that G contains at least $\lambda(G)$ many edge-disjoint u-v paths. Since they're edge-disjoint, at most one of them uses the edge e and so $G - e$ contains at least $\lambda(G) - 1$ many edge-disjoint u-v paths. Thus, Menger's theorem for edge-connectivity implies that $\lambda(G - e) \geq \lambda(G) - 1$.

2. Technically, I shouldn't allow the following proof since we relied on this fact in order to prove Menger's theorem... But I'll allow it just this once.

   Since $G - e$ is a spanning subgraph of G, Problem 2 implies that $\kappa(G) \geq \kappa(G - e)$; thus we need only prove that $\kappa(G - e) \geq \kappa(G) - 1$.

   Fix any $u \neq v \in V(G) = V(G - e)$. Menger's theorem for vertex-connectivity tells us that G contains at least $\kappa(G)$ many internally-disjoint u-v paths. Certainly internally-disjoint paths are also edge-disjoint and so at most one of them uses the edge e which implies that $G - e$ contains at least $\kappa(G) - 1$ many internally-disjoint u-v paths. Thus, Menger's theorem for vertex-connectivity yields $\kappa(G - e) \geq \kappa(G) - 1$.

□

Problem 4 (1 pts)**.** For each non-negative integer k, find an example of a graph G with $\kappa(G) = \lambda(G) = 1$, yet there is some vertex $v \in V(G)$ such that $\kappa(G - v) = \lambda(G - v) = k$.

That is to say, the natural analogue of Problem 3 fails when deleting vertices instead of edges.

Solution. Let G be a copy of K_{k+1} with an extra leaf attached to one of the vertices. Formally, G has vertex-set $V(G) = \{u_1, \ldots, u_{k+1}, v\}$ where $G[\{u_1, \ldots, u_{k+1}\}] \cong K_{k+1}$ and the unique neighbor of v is u_1. Now, G is connected and is not a copy of K_1 and so $\lambda(G) \geq \kappa(G) \geq 1$. Additionally, $u_1 v$ is a cut-edge of G since v is an isolated vertex of $G - u_1 v$; hence $1 \geq \lambda(G) \geq \kappa(G)$. In conclusion, $\lambda(G) = \kappa(G) = 1$.

Now, consider $G - v$, which is a copy of K_{k+1}. Thus, $\kappa(G - v) = \kappa(K_{k+1}) = k$ and $\lambda(G - v) = \lambda(K_{k+1}) = k$. □

Problem 5 (2 pts)**.** Let G be a graph. The *kth power of* G is the graph G^k which has the same vertex-set as G and $uv \in E(G^k)$ iff $d_G(u, v) \leq k$.

Prove that if G is a connected graph on at least $k + 1$ vertices, then G^k is k-connected.

(You don't have to turn this is, but you should convince yourself that it's true: G^k is a clique if and only if $\text{diam}(G) \leq k$.)

Solution. [#1] Set $V = V(G) = V(G^k)$.

Suppose that G has n vertices ($n \geq k+1$ by assumption). If G^k is a clique, then we have $\kappa(G^k) = n - 1 \geq (k+1) - 1 = k$ and so G^k is k-connected as desired. Thus, we may suppose that G^k is not a clique.

Thus, suppose that $U \subseteq V$ is a vertex-cut of G^k; we must show that $|U| \geq k$, so suppose for the sake of contradiction that $|U| \leq k-1$. Since U is a vertex-cut of G^k we know that $G^k - U$ is disconnected; thus we may partition $V(G^k - U) = V \setminus U = A \sqcup B$ with A, B non-empty such that there are no edges of $G^k - U$ crossing between A and B. Now, G is connected, so there is an a-b path in G for all $a \in A$ and $b \in B$. Consider a pair $a \in A$, $b \in B$ such that $d_G(a,b)$ is minimum. Let $(a = v_0, \ldots, v_s = b)$ be a a-b geodesic in G. If $s \leq k$, then $d_G(a,b) \leq k$ and so $ab \in E(G^k)$ (a contradiction); thus we may suppose that $s \geq k+1$. Thus, $|\{v_1, \ldots, v_{s-1}\}| = s - 1 \geq k > |U|$. Therefore, there is some $i \in [s-1]$ such that $v_i \notin U$: thus, either $v_i \in A$ or $v_i \in B$. In the former case, $v_i \in A$ has $d_G(v_i, b) = s - i < s = d_G(a,b)$; contradiction. In the latter case, $v_i \in B$ has $d_G(a, v_i) = i < s = d_G(a,b)$; contradiction. □

Solution. [#2] We seek to apply Menger's theorem, so we must prove that there are at least k internally-disjoint paths between any pair of distinct vertices in G^k.

We begin with a lemma.

Lemma 1. *Suppose that G is connected and has at least n vertices. If G_1 is any connected subgraph of G with at most n vertices, then we can find another connected subgraph G_2 of G with exactly n vertices such that G_1 is a subgraph of G_2.*

Proof. Suppose that $|V(G_1)| = s \leq n$; we prove the claim by induction on $n - s$.

As a base-case, suppose that $n - s = 0$, so $s = n$ and so we can simply take $G_2 = G_1$.

Now suppose that $n - s \geq 1$, so $1 \leq s \leq n-1$. Then $V(G_1)$ and $V(G) \setminus V(G_1)$ are both non-empty and partition $V(G)$. Since G is connected, there must be some edge $uv \in E(G)$ with $u \in V(G_1)$ and $v \in V(G) \setminus V(G_1)$. Let G_2' be the graph formed by appending the edge uv to G_1. Certainly G_2' is still a subgraph of G and G_1 is a subgraph of G_2'. Furthermore, since G_1' is connected, certainly G_2' is connected. Notice that $|V(G_2')| = s + 1 \leq n$ and so the induction hypothesis guarantees that there is a connected subgraph G_2 of G with exactly n vertices such that G_2' is a subgraph of G_2. Of course, G_1 is also a subgraph of this G_2. □

With that out of the way, we can find our desired paths. Fix any $u \neq v \in V(G)$; we need to show that there are at least k internally-disjoint paths between u and v in G^k. We break into cases depending on $d_G(u,v)$ (which we know is finite since G is connected).

Case 1: $d_G(u,v) \leq k$. Let $(u = v_0, \ldots, v_s = v)$ be a u-v geodesic in G, so $s \leq k$. This path is a connected, $(s+1)$-vertex connected subgraph of G, so, since G has at least $k+1$ vertices by assumption, Lemma 1 guarantees that we can find a connected, $(k+1)$-vertex subgraph G_2 of G which contains this path. Since G_2 is connected and has $k+1$ vertices, we know that $\operatorname{diam}(G_2) \leq k$. In particular, for any $x, y \in V(G_2)$, we have $d_{G_2}(x,y) \leq k$ and so $xy \in E(G_2^k)$. In particular, $G_2^k \cong K_{k+1}$. Since K_{k+1} is k-connected, we can find k internally-disjoint u-v paths in G_2^k. Finally, certainly G_2^k is a subgraph of G^k and so we have succeeded in finding our k internally-disjoint u-v paths.

Case 2: $d_G(u,v) > k$. Again, let $(u = v_0, \ldots, v_s = v)$ be a u-v geodesic in G, so $s > k$. Since this is a geodesic, we know that $d_G(v_i, v_j) = |i - j|$ for all $i, j \in \{0, \ldots, s\}$. In particular, $v_i v_j \in E(G^k)$ whenever $i \neq j$ and $|i - j| \leq k$. Since $s > k$, we can write $s = qk + r$ where q is a positive integer and $r \in [k]$. For each $i \in [k]$, define

$$P_i = \begin{cases} (u = v_0, v_i, v_{k+i}, \dots, v_{qk+i}, v_s = v) & \text{if } i < r, \\ (u = v_0, v_i, v_{k+i}, \dots, v_{(q-1)k+i}, v_s = v) & \text{if } i \geq r. \end{cases}$$

By the earlier comment, each of $P_1, \dots, P_k$ are u-v paths in G^k since all consecutive indices in each of them differ by at most k. Furthermore, since residue classes modulo k are disjoint, these are k internally-disjoint u-v paths in G^k as needed. □