These solutions are from https://mathematicaster.org/teaching/graphs2022/sol-hw8.pdf

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

We say that a graph is *even-regular* if every vertex has even degree, and we say that a graph is *odd-regular* if every vertex has odd degree.

Problem 1 (2pts)**.** In the second week of class (01-27), we used the handshaking lemma to prove the following fact: If G is a connected even-regular graph, then G has no bridges.

Give an alternative proof of this fact using what we now know about Eulerian circuits.

Solution. Note that if G has no edges, then certainly G has no bridges, so we may suppose G has at least one edge.

Fix any edge $e \in E(G)$; we must show that $G - e$ is also connected. Since G is connected and even-regular, we know that G contains an Eulerian circuit; label the vertices of this circuit as $C = (v_0, v_2, \dots, v_m = v_0)$ (G is simple, so we can get away with this). Since the edge e is traversed exactly once in this circuit, we may suppose, without loss of generality, that $e = v_{m-1}v_m$. Now, consider the walk $(v_0, v_2, \dots, v_{m-1})$, which still sees all vertices of G since $v_m = v_0$. Since this walk sees all vertices of $G - e$, we conclude that $G - e$ is connected. $\square$

Problem 2 (2 + 2 pts)**.** A digraph D is said to be *oriented* if $(u, v) \in E(D) \implies (v, u) \notin E(D)$, i.e. D has no directed cycles of length 1 or 2.

For a digraph D with no loops, the *underlying simple graph* of D is the (simple) graph G with $V(G) = V(D)$ and $uv \in E(G)$ if and only if $(u, v) \in E(D)$ or $(v, u) \in E(D)$. In other words, the underlying simple graph is formed by simply forgetting about the directions of the edges.

For a (simple) graph G, an *orientation* of G is an oriented digraph whose underlying simple graph is G. In other words, an orientation of G is formed by assigning a direction to each edge of G. Note that a graph generally has many orientations.

1. Prove that if G is a (simple) even-regular graph, then G has an orientation wherein $\deg^+ v = \deg^- v$ for all vertices v.

2. Prove that if G is any (simple) graph, then G has an orientation wherein $|\deg^+ v - \deg^- v| \le 1$ for all vertices v.

   You are free use part 1 as a black-box even if you haven't proved it.

Solution.

1. To begin, we may suppose that G is connected since we could apply the result to each individual connected component of G otherwise.

   Thus, G is connected and even-regular, so G contains an Eulerian circuit; label the vertices of this circuit as $C = (v_0, v_2, \dots, v_m = v_0)$. From C, we build the desired orientation D — informally, we orient the edges of G based on how they are traversed in C. Formally, for an edge $e \in E(G)$, we know that there is a *unique* $i \in \{0, \dots, m-1\}$ such that $e = v_iv_{i+1}$; add the directed edge (v_i, v_{i+1}) to D. Observe that D is an orientation of G since each edge is traversed exactly once in C and so each edge was given a unique direction.

We now show that D has the property that $\deg^+ v = \deg^- v$ for all $v \in V(D) = V(G)$ — this is essentially identical to what we did in class to prove that Eulerian graphs have all degrees even. Fix any $v \in V(D)$ and define $I = \{i \in \{0, \ldots, m-1\} : v_{i+1} = v\}$ and $O = \{i \in \{0, \ldots, m-1\} : v_{i-1} = v\}$ where all indices here are computed modulo m. By construction, $\deg^- v = |I|$ and $\deg^+ v = |O|$. Furthermore, $|I| = |O|$ as witnessed by the bijection $i \mapsto i + 2 \pmod{m}$, and so D is indeed our desired orientation.

2. Let $U \subseteq V(G)$ be the set of all odd-degree vertices of G and form a new graph G' by introducing a new vertex x to G which is adjacent to all vertices in U. Observe that if $u \in U$, then $\deg_{G'} u = \deg_G u + 1$, which is even since $\deg_G u$ is odd. Also, $\deg_{G'} x = |U|$, which is even thanks to the handshaking lemma (even number of odd degrees). Finally, if $v \in V(G) \setminus U$, then $\deg_{G'} v = \deg_G v$, which is even. Therefore, G' is even-regular.

   We can thus apply part 1 to G' to find an orientation D' such that $\deg^+_{D'} v = \deg^-_{D'} v$ for all $v \in V(G') = V(G) \cup \{x\}$.

   Now, consider the digraph $D = D' - x$, which is an orientation of G. If $v \in V(G) \setminus U$, then $\deg^\pm_D v = \deg^\pm_{D'} v$ and so $\deg^+_D v = \deg^-_D v$.

   On the other hand, consider any $u \in U$. If $(x, u) \in E(D')$, then $\deg^-_D u = \deg^-_{D'} u - 1$ and $\deg^+_D u = \deg^+_{D'} u$. If $(u, x) \in E(D')$, then $\deg^+_D u = \deg^+_{D'} u - 1$ and $\deg^-_D u = \deg^-_{D'} u$. In either case, we have $|\deg^+_D u - \deg^-_D u| = 1$.

$\square$

Problem 3 (2 + 2 pts)**.** For graphs G, H, the *Cartesian product* of G and H is the graph $G \,\square\, H$ which has vertex set $V(G) \times V(H)$ and $\{(u_1, v_1), (u_2, v_2)\} \in E(G \,\square\, H)$ if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $u_1 u_2 \in E(G)$ and $v_1 = v_2$.[1]

Suppose that G and H are any graphs.

1. Prove that $G \,\square\, H$ is connected if and only if both G and H are connected.

2. Prove that $G \,\square\, H$ is Eulerian if and only if both G and H are connected and also:

   (a) Both G and H are even-regular, or

   (b) Both G and H are odd-regular.

   You are free to use part 1 as a black-box even if you haven't proved it.

Solution.

1. ($\Leftarrow$) Consider any $(u_1, v_1), (u_2, v_2) \in V(G \,\square\, H)$. Since G is connected, there is a u_1-u_2 path in G, call it $(u_1 = w_1, \ldots, w_k = u_2)$. Then, $\big((w_1, v_1), (w_2, v_1), \ldots, (w_k, v_1)\big)$ is a (u_1, v_1)-(u_2, v_1) path in $G \,\square\, H$. Similarly, H is connected, so there is a v_1-v_2 path in H, call it $(v_1 = z_1, \ldots, z_\ell = v_2)$. Then $\big((u_2, z_1), \ldots, (u_2, z_\ell)\big)$ is a (u_2, v_1)-(u_2, v_2) path in $G \,\square\, H$. By concatenating these two paths, we obtain a (u_1, v_1)-(u_2, v_2) walk and thus know that $G \,\square\, H$ is connected.

[1] N.b. We like the notation $\square$ here since $K_2 \,\square\, K_2 \cong C_4$, which looks like a $\square$. Note that your book uses $\times$ in place of $\square$; this is okay, but not desirable since generally $\times$ denotes a different graph product known as the categorical product, in which $K_2 \times K_2 \cong K_2 \sqcup K_2$, which can be made to look like an $\times$.

($\Rightarrow$) We prove the contrapositive, so we must show that if G or H is disconnected, then $G \square H$ is disconnected. Note that $G \square H \cong H \square G$ as witnessed by the isomorphism $(u, v) \mapsto (v, u)$, so, without loss of generality, we may suppose that G is disconnected.

Since G is disconnected we can partition $V(G) = A \sqcup B$ with A, B non-empty and no edge of G crosses between A and B. Set $A' = A \times V(H)$ and $B' = B \times V(H)$, so $V(G \square H) = A' \sqcup B'$ with A', B' non-empty. We claim that there is no edge of $G \square H$ which crosses between A' and B', which will imply that $G \square H$ is disconnected. Consider any $(a, v_1) \in A' = A \times V(H)$ and $(b, v_2) \in B' = B \times V(H)$; we must show that (a, v_1) is not adjacent to (b, v_2) in $G \square H$. Note that $a \in A$ and $b \in B$ and so $a \neq b$ since A and B are disjoint; thus the only way that $\{(a, v_1), (b, v_2)\} \in E(G \square H)$ is if $v_1 = v_2$ and $ab \in E(G)$, which is impossible since G has no edges crossing between A and B.

2. To begin, for any $(u, v) \in V(G \square H)$, we find that

$$N_{G\square H}(u, v) = \big\{(u', v) : u' \in N_G(u)\big\} \sqcup \big\{(u, v') : v' \in N_H(v)\big\}.$$

Thus, $\deg_{G\square H}(u, v) = \deg_G u + \deg_H v$, which we will use throughout.

($\Rightarrow$) Since G and H are connected, we know that $G \square H$ is connected thanks to part 1; thus to show that $G \square H$ is Eulerian, we must show that every vertex has even degree. Consider any $(u, v) \in V(G \square H)$; we know that $\deg_{G\square H}(u, v) = \deg_G u + \deg_H v$. By assumption, either G and H are both even-regular or both G and H are odd-regular, and so either $\deg_G u$ and $\deg_H v$ are both even or are both odd. In either case $\deg_{G\square H}(u, v)$ is even, as needed.

($\Leftarrow$) Since $G \square H$ is Eulerian is connected, we know that $G \square H$ is connected; hence both G and H are connected thanks to part 1. Thus, we must show that either G and H are both even-regular or that both are odd-regular.

Now, since $G \square H$ is Eulerian, we know that $\deg_{G\square H}(u, v)$ is even for all $(u, v) \in V(G \square H)$. In particular, $\deg_G u + \deg_H v$ is even for all $u \in V(G)$ and $v \in V(H)$. We conclude that $\deg_G u$ and $\deg_H v$ have the same parity for every $u \in V(G)$ and every $v \in V(H)$. Thus, either $\deg_G u$ and $\deg_H v$ are even for all $u \in V(G)$ and $v \in V(H)$ or $\deg_G u$ and $\deg_H v$ are odd for all $u \in V(G)$ and $v \in V(H)$. In other words, either both G and H are even-regular, or both are odd-regular.

$\square$