These solutions are from https://mathematicaster.org/teaching/graphs2022/sol-hw9.pdf

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (1 + 2 pts)**.** Fix any integer $n \geq 3$.

1. Construct an n-vertex graph G with $\binom{n-1}{2} + 1$ many edges such that G is *not* Hamiltonian. (Note: You must construct such a graph for *every* $n \geq 3$.)
2. Prove that if G is an n-vertex graph with at least $\binom{n-1}{2}+2$ many edges, then G is Hamiltonian.

Solution.

1. Form G by attaching a leaf to K_{n-1}. Formally, G has vertex-set $V(G) = \{v_1, \dots, v_{n-1}, u\}$, where $G[\{v_1, \dots, v_{n-1}\}] \cong K_{n-1}$ and the unique neighbor of u is v_1. Since $\deg u = 1$, G cannot be Hamiltonian since Hamiltonicitity requires minimum degree at least two (though G does contain a Hamiltonian path). Furthermore, $|E(G)| = 1 + |E(G[\{v_1, \dots, v_{n-1}\}])| = 1 + |E(K_{n-1})| = 1 + \binom{n-1}{2}$ as requested.
2. Fix any non-adjacent vertices u, v (if no such pair exists, then we're done since $n \geq 3$). Set $G' = G - \{u, v\}$ (deleting vertices here). Since u and v are non-adjacent, we find that $|E(G)| = |E(G')| + \deg u + \deg v$. On the other hand, G' has $n-2$ vertices and so $|E(G')| \leq \binom{n-2}{2}$. Putting these together, we have

$$\binom{n-1}{2} + 2 \leq |E(G)| = |E(G')| + \deg u + \deg v \leq \binom{n-2}{2} + \deg u + \deg v$$
$$\implies \deg u + \deg v \geq \binom{n-1}{2} + 2 - \binom{n-2}{2} = n.$$

   Thus, G satisfies Ore's condition and so G is Hamiltonian.

□

Problem 2 (0.5 + 0.5 pts)**.** Recall that K_{n_1,n_2,n_3} is the complete tripartite graph with parts of sizes n_1, n_2, n_3.

Fix any positive integer n.

1. Prove that $K_{n,2n,3n}$ is Hamiltonian.
2. Prove that $K_{n,2n,3n+1}$ is not Hamiltonian.

Solution.

1. Call the parts A, B, C where $|A| = n$, $|B| = 2n$ and $|C| = 3n$.

   We observe that for $a \in A, b \in B, c \in C$, we have $\deg a = 5n$, $\deg b = 4n$ and $\deg c = 3n$. Thus $\delta(K_{n,2n,3n}) \geq 3n = \frac{1}{2}|V(K_{n,2n,3n})|$ and so Dirac's condition implies that $K_{n,2n,3n}$ is Hamiltonian since certainly $6n \geq 3$.

2. Call the parts A, B, C where $|A| = n$, $|B| = 2n$ and $|C| = 3n+1$.

   Since C is an independent set, we observe that $\text{comp}(K_{n,2n,3n+1} - (A \cup B)) = |C| = 3n+1$. However, $|A \cup B| = 3n < 3n+1$ and so $K_{n,2n,3n+1}$ cannot be Hamiltonian.

□

Problem 3 (2pts)**.** Let G be a graph on $n \geq 4$ vertices with the property that $N(u) \cup N(v) \supseteq V(G) \setminus \{u, v\}$ for every $u \neq v \in V(G)$. Prove that G is Hamiltonian.

(Hint: $|A \cup B| = |A| + |B| - |A \cap B|$ for finite sets A, B.)

(Hint: The problem and first hint suggest attempting a proof similar to our proof of Bondy–Chvátal, i.e. extending a Hamiltonian path to a Hamiltonian cycle. While such a proof is possible, it's more difficult than a more direct proof using only Ore's condition. This is just my opinion and maybe you disagree; I just don't want to lead you down the wrong path.)

Solution. [#1] We verify Ore's condition. Fix any non-adjacent $u \neq v \in V(G)$ (if there is no such pair, then we are done since $n \geq 3$). We need to show that $\deg u + \deg v \geq n$. Suppose for the sake of contradiction that $\deg u + \deg v \leq n-1$. Now,

$$\begin{aligned} n-2 = |V(G) \setminus \{u,v\}| &\leq |N(u) \cup N(v)| = |N(u)| + |N(v)| - |N(u) \cap N(v)| \\ &= \deg u + \deg v - |N(u) \cap N(v)| \leq n-1 - |N(u) \cap N(v)|, \end{aligned}$$

and so $|N(u) \cap N(v)| \leq 1$, i.e. u and v have at most one common neighbor. Since $n \geq 4$, this implies that there must be some $x \in V(G) \setminus \{u,v\}$ with $x \notin N(u) \cap N(v)$. Without loss of generality, we may suppose that $x \notin N(v)$.

Consider the vertices x and u. By assumption, $v \in N(u) \cup N(x)$ since $v \in V(G) \setminus \{x, u\}$. However, $v \notin N(u)$ since u and v were assumed to not be adjacent and also $v \notin N(x)$ since $x \notin N(v)$. Therefore $v \notin N(u) \cup N(x)$; a contradiction. □

Solution. [#2] We follow the ideas in our proof of Bondy–Chvátal.

First, we show that G has a Hamiltonian path. Indeed, suppose that $P = (v_1, \ldots, v_k)$ is a maximum-length path in G. Notice that $k \geq 2$ since $n \geq 4$ and an independent set of size at least 3 cannot possibly satisfy the assumption; in particular $v_1 \neq v_k$.

If $k = n$ then we're done, so suppose for the sake of contradiction that $k < n$. Since $k < n$, there is some $x \in V(G) \setminus V(P)$. Since $v_1 \neq v_k$, we thus have $x \in N(v_1) \cup N(v_k)$ by assumption. If $x \in N(v_k)$, then $(v_1, \ldots, v_k, x)$ is a strictly longer path than P; a contradiction. If $x \in N(v_1)$, then $(x, v_1, \ldots, v_k)$ is a strictly longer path than P; a contradiction. Thus, $k = n$ and so G does indeed have a Hamiltonian path.

Now that we know that G has a Hamiltonian path, let $(v_1, \ldots, v_n)$ be such a path. If also $v_1 v_n \in E(G)$, then we have found a Hamiltonian cycle, so suppose this is not the case.

Observe that $n \geq 4$ and so $v_2, v_3 \notin \{v_1, v_n\}$. Identically to our proof of Bondy–Chvátal, if $v_1 v_3, v_2 v_n \in E(G)$, then we have found a Hamiltonian cycle, so we show that this is indeed the case which will conclude the proof. If $v_1 v_3 \notin E(G)$, then, since $v_1 v_n \notin E(G)$, we have $v_1 \in V(G) \setminus \{v_3, v_n\}$ and $v_1 \notin N(v_3) \cup N(v_n)$; a contradiction. Symmetrically, if $v_2 v_n \notin E(G)$, then, since $v_1 v_n \notin E(G)$, we have $v_n \in V(G) \setminus \{v_1, v_2\}$ and $v_n \notin N(v_1) \cup N(v_2)$; a contradiction. □

Solution. [#3] This is probably the easiest proof, but I didn't think of it until right before I posted these solutions (I was distracted by the nice application of Ore's conditions in the first solution).

Fix any $v \in V(G)$; we claim that $\deg v \geq n-2$. If not, then $\deg v \leq n-3$ and so $|\{v\} \cup N(v)| \leq n-2$. But then we can find $x \neq y \in V(G) \setminus (\{v\} \cup N(v))$. Now, $v \in V(G) \setminus \{x,y\}$ yet $x, y \notin N(v) \implies v \notin N(x) \cup N(y)$; a contradiction.

Thus, we have $\delta(G) \geq n-2$, and also $n-2 \geq n/2$ since $n \geq 4$. Therefore $\delta(G) \geq n/2$ and $n \geq 3$, so G satisfies Dirac's condition and is thus Hamiltonian. □

Problem 4 (2pts)**.** Let G be a graph on n vertices. In class, we showed that $\alpha'(G) + \beta'(G) = n$ provided G has no isolated vertices; this exercise exists to establish the natural (and easier to prove) vertex-version of this fact.

Recall that the independence number of G, denoted by $\alpha(G)$, is the size of a largest independent set of G. Recall also that the vertex-cover number of G, denoted by $\beta(G)$, is the size of a smallest vertex-cover of G.

Prove that $\alpha(G) + \beta(G) = n$ for any n-vertex graph G.

(Note: You don't need to know anything about matchings to prove this.)

Solution. We show first that $\alpha(G) + \beta(G) \leq n$. To do so, let $A \subseteq V$ be a maximum independent set in G, so $|A| = \alpha(G)$. Since A is an independent set, no edge of G can have both end-points in A and so every edge of G has at least one end-point in $V \setminus A$. In other words, $V \setminus A$ is a vertex-cover of G and so $\beta(G) \leq |V \setminus A| = n - \alpha(G) \implies \alpha(G) + \beta(G) \leq n$.

Next, we show that $\alpha(G) + \beta(G) \geq n$. To do so, let $B \subseteq V$ be a minimum vertex-cover of G, so $|B| = \beta(G)$. Since B is a vertex-cover, every edge of G has at least one end-point in B. Therefore, no edge has both end-points in $V \setminus B$ and so $V \setminus B$ is an independent set in G. Therefore, $\alpha(G) \geq |V \setminus B| = n - \beta(G) \implies \alpha(G) + \beta(G) \geq n$. □

Problem 5 (2 pts)**.** We have an $m \times n$ matrix $M \in \{0,1\}^{m \times n}$. We refer to the columns and rows of this matrix as *lines* (so a line is either one of the m rows or one of the n columns of M). Prove that the minimum number of lines needed to contain all 1's of M is precisely the maximum number of 1's in M that one can select so that no two of these selected 1's live in any common line.

Solution. This is just a restatement of Kőnig's theorem in non-graph-theoretic language.

We build a bipartite graph G with parts $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$ where $a_i b_j \in E(G)$ if and only if $M_{ij} = 1$. In other words, A corresponds to the rows of M, B corresponds to the columns of M and edges correspond to the 1's in M. In particular, the lines of M correspond to all of $V(G)$ and a line contains a particular 1 if and only if the corresponding vertex is an end-point of the corresponding edge.

With this correspondence we have:

- A collection of lines which contain all 1's of M corresponds to a vertex-cover of G. Thus, the fewest number of lines necessary to contain all 1's is precisely $\beta(G)$.
- A collection of 1's in M with no two in a common line corresponds to a matching in G. Thus, the maximum number of such 1's in M is precisely $\alpha'(G)$.

Since G is bipartite, Kőnig tells us that $\beta(G) = \alpha'(G)$ and so the claim follows. □